Complementary Information and Learning Traps*

Annie Liang[†] Xiaosheng Mu[‡]

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Abstract

We develop a model of social learning from complementary information: Short-lived agents sequentially choose from a large set of (flexibly correlated) information sources for prediction of an unknown state, and information is passed down across periods. Will the community collectively acquire the best kinds of information? Long-run outcomes fall into one of two cases: (1) efficient information aggregation, where the community eventually learns as fast as possible; (2) "learning traps," where the community gets stuck observing suboptimal sources and information aggregation is inefficient. Our main results identify a simple property of the underlying informational complementarities that determines which occurs. In both regimes, we characterize which sources are observed in the long run and how often. These results hold both for persistent and for slowly changing states.

1 Introduction

We consider social learning from complementary information. Consider for example researchers studying the effect of sleep loss on depression. There are many studies relevant to this question, and if research were conducted simultaneously, we might ask what combination of studies would shed most light on this question. But researchers choose what information to acquire at different times, and their choices are influenced not by a planner, but by the history of research—in particular, the study that is most informative right now depends on what has been previously done. Informational complementarities are critical in

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[†]University of Pennsylvania

[‡]Cowles Foundation and Columbia University

these decisions: For example, if past researchers have developed methods for measuring the hormone cortisol, but not the neurotransmitter dopamine, lab researchers are more likely to obtain measurements of the former.[1]

Whether informational complementarities push research in a socially beneficial direction is not clear from intuition alone. Take for example the complementarity between improved methods for measuring cortisol, and measurement of cortisol. If cortisol is the most informative quantity to be measuring, this complementarity creates a positive externality, pushing researchers to cycle through theory and measurement towards an informative body of work. In contrast, if study and measurement of neurotransmitters such as dopamine would be more informative, we might worry that the informational complementarities described above constitute a distraction. Thus, understanding the externality imposed by informational complementarities is important for identifying long-run information acquisition patterns, and also for identifying interventions for shaping these acquisitions.

Our framework is a social learning model where agents, indexed by discrete time, acquire information and take actions (prediction of a payoff-relevant state). We depart from the classic sequential learning model (Banerjee, 1992; Bikhchandani, Hirshleifer and Welch, 1992; Smith and Sorenson, 2000) in two key ways: First, we suppose that all information is public, so that predictions are based on the history of signal realizations so far. This departure turns off the inference problem essential to the existence of cascades in standard herding models. Second, we assume endogenous information acquisition—specifically, agents choose from a large number of information sources, each associated with a signal about a payoff-relevant state.[2] The available sources provide different (noisy) linear combinations of the payoff-relevant state and a set of "confounding" variables.[3] We develop a notion for *complementary sets* of sources inspired by Borgers, Hernando-Veciana and Krahmer (2013), and allow for the presence of many, overlapping, complementary sets of sources.

Besides our example above, informational complementarities appear in many other settings—for example, different news sources may cover complementary topics; team members may have expertise on different aspects of a project (as in Chade and Eeckhout (2018)); and market participants may have dispersed views and opinions that can be usefully aggre-

[1] Cortisol can be straightforwardly measured in saliva, while direct measurement of dopamine, a neurotransmitter in the brain, currently requires invasive procedures such as placing electrodes in the brain.

[2] Here we build on Burguet and Vives (2000), Mueller-Frank and Pai (2016), and Ali (2018), who introduce endogenous information acquisition to a classic social learning setting. Relative to this work, our paper considers choice from a fixed set of information sources (with a capacity constraint), in contrast to choice from a flexible set of information sources (with a cost on precision).

[3] There is a large body of papers that model information as flexibly correlated Gaussian signals as we do, for example Angeletos and Pavan (2007), Hellwig and Veldkamp (2009), Myatt and Wallace (2012), Lambert, Ostrovsky and Panov (2018) among others. These prior works focus on simultaneous information acquisition by strategic players.

gated (as in Goldstein and Yang (2015)). Correspondingly, there is an extensive literature on informational complementarities (Milgrom and Weber, 1982*a*,*b*; McLean and Postlewaite, 2002; Borgers, Hernando-Veciana and Krahmer, 2013; Chen and Waggoner, 2016), although the prior work has focused on one-time information acquisitions. In dynamic settings, informational complementarities adopt a new role: They structure how past information acquisitions influence the value of information sources for later agents. Our main results relate long-run aggregation of information to these underlying informational complementarities. Specifically, we demonstrate that the size of the smallest complementary set is critical for separating two very different outcomes: guaranteed *efficient information aggregation*—past information pushes agents towards the best kinds of information—and *learning traps*—early suboptimal information acquisitions propagate across time, and there are persistent inefficiencies in information gathering. Our focus on the rate of information aggregation builds on Vives (1992), Golub and Jackson (2012), Hann-Caruthers, Martynov and Tamuz (2017), and Harel et al. (2018) among others.[4]

As a benchmark, we begin by deriving the *optimal* long-run frequency of signal acquisitions. These correspond to the choices that maximize information revelation about the payoff-relevant state, and also to the choices that maximize a discounted sum of agent payoffs (in a patient limit). We show that these optimal acquisitions eventually concentrate on a "best" complementary set of signals.

Whether society's acquisitions converge to this optimal long-run frequency depends critically on the size of the smallest complementary set of signals. We show that from any prior, information acquisitions eventually concentrate on *some* complementary set of signals. If the smallest complementary set is at least size K, where K is the number of unknowns (including the payoff-relevant state and all confounding variables), agents will come to discover the best overall set of signals. The key intuition here refers back to an observation made in Sethi and Yildiz (2016): An agent who repeatedly observes a source confounded by an unknown parameter learns *both* about the payoff-relevant state and also about the confounding parameter, and hence improves his interpretation of all sources confounded by the same parameter. If K sources are repeatedly observed, then agents will acquire information that (collectively) reveals all of the unknowns, eventually evaluating *all* sources by a prior-independent asymptotic criterion. This allows them to identify the best set of sources.

In contrast, if some complementary set consists of fewer than K sources, then agents can persistently undervalue sources that provide information confounded by the remaining variables, and long-run learning may be inefficient. Our second main result says that any complementary set with fewer than K sources creates a "learning trap" under some set of

[4]There is a large literature on the inefficiencies of information gathering in strategic interactions; see for example Angeletos and Pavan (2007) and Myatt and Wallace (2012).

prior beliefs.

We next study interventions for breaking learning traps. We show that policymakers can restore efficient information aggregation by providing sufficiently many kinds of free information, or by reshaping the reward structure so that agents' predictions are based on information acquired over many periods.

The final part of our paper considers the welfare losses associated with learning traps. We show that the rate of information aggregation can be arbitrarily slow (relative to the efficient benchmark), and payoffs can be arbitrarily inefficient when we consider the *ratio* of achieved and feasible payoffs in a patient limit. However, because the payoff-relevant state is persistent across time, agents eventually learn its value even while in a learning trap. Thus payoff losses are negligible when measured as the *difference* of achieved and feasible payoffs, both of which vanish in the patient limit. We demonstrate next that in nearby models in which states are not fully persistent, this conclusion fails and average payoff difference can also be arbitrarily large.

To show this, we consider a generalization of the model where the state vector is changing over time. This is a technically challenging setting to analyze, and correspondingly prior work is very limited.[5] We consider a sequence of autocorrelated models that converge to our main model, and show that signal sets constituting potential learning traps remain potential learning traps for autocorrelation sufficiently close to 1. Welfare losses in learning traps (measured either by the payoff ratio or payoff difference) can be arbitrarily large when the state is nearly (but not perfectly) persistent.

The main technical difficulty in analyzing this extension is the failure of signal exchangeability. Unlike in the main model, posterior variance about the payoff-relevant state can no longer be expressed as a function of counts for how often each source has been observed. Instead, we work with the covariance matrix of the entire state vector and study a controlled dynamical system. By constructing a Lyapunov function in this matrix that is monotonic over time, we are able to bound the speed of learning. We do not pursue a full characterization of the autocorrelated model, although this is an interesting question for future work.

Besides the papers mentioned above, this paper builds on a recent literature that studies choice from a discrete and fixed set of information sources—see for example Che and Mierendorff (2017) and Mayskaya (2017), who study choice between two Poisson sources, and Sethi and Yildiz (2016) and Fudenberg, Strack and Strzalecki (2018), who study choice between multiple Gaussian sources. For the most part, these models have not allowed for flexible

[5] Moscarini, Ottaviani and Smith (1998), Frongillo, Schoenebeck and Tamuz (2011), Vivi Alatas and Olken (2016), and Dasaratha, Golub and Hak (2018) are the only social learning settings with a dynamic state that we are aware of.

correlations across the available kinds of information, and thus preclude complementarities across sources.

Sethi and Yildiz (2017) considers an informational environment similar to ours. The sources of information are interpreted as people, who themselves listen to other sources (people) over time, and the focus is on correlation structures emerging from disjoint communities of individuals with community-specific bias terms. Sethi and Yildiz (2017) shows that individuals can exhibit homophily in the long run, listening only to other individuals from the same community; this phenomenon is related to the observation of learning traps in the present paper.

Finally, this paper contributes to a broader question regarding dynamic information acquisition from flexibly correlated information sources. In our earlier paper Liang, Mu and Syrgkanis (2017), we considered a special case of the environment studied here, where the set of available sources consisted of a single complementary set. This restriction allowed us to (under some conditions) fully characterize the optimal information acquisition strategy for any discount factor, including the myopic behavior studied in the current paper.

2 Setup

Informational Environment. There are K persistent unknown states: a real-valued payoff-relevant state ω and $K-1$ real-valued confounding states $b_1, \dots, b_{K-1}$. We assume that the state vector $\theta := (\omega, b_1, \dots, b_{K-1})'$ follows a multivariate normal distribution $\mathcal{N}(\mu^0, \Sigma^0)$ where $\mu^0 \in \mathbb{R}^K$, and the prior covariance matrix Σ^0 has full rank.[6,7]

There are N (fixed) *kinds* or *sources* of information available at each discrete period $t \in \mathbb{Z}_+$. Observation of source i in period t produces a realization of the random variable

$$X_i^t = \langle c_i, \theta \rangle + \epsilon_i^t, \quad \epsilon_i^t \sim \mathcal{N}(0,1)$$

where $c_i = (c_{i1}, \dots, c_{iK})'$ is a vector of constants, and the error terms ϵ_i^t are independent from each other and across periods. Normalizing these error terms to have unit variance is without loss of generality, since the coefficients c_i are unrestricted. We will often drop the time indices on the random variables, associating $X_i = \langle c_i, \theta \rangle + \epsilon_i$ with source i and understanding that the error term is independently realized with each new observation.

The payoff-irrelevant states $b_1, \dots, b_{K-1}$ produce correlations across the sources, and can be interpreted for example as:

[6] The full rank assumption is without loss of generality: If there is linear dependence across the states, the model can be reduced to a lower dimensional state space that satisfies full rank.

[7] Unless otherwise indicated, vectors in this paper are column vectors.

- *Confounding explanatory variables*: Observation of signal i produces the (random) outcome $y = \omega c_i^1 + b_1 c_i^2 + \ldots b_{K-1} c_i^K + \epsilon_i$, which depends linearly on an observable characteristic vector c_i. For example, y might be the average incidence of depression in a group of individuals with characteristics c_i. The state of interest ω is the coefficient on a given characteristic c_i^1 (i.e. average hours of sleep), and the payoff-irrelevant states are the unknown coefficients on the auxiliary characteristics $c_i^2, \ldots, c_i^K$. Different sources represent subpopulations with different characteristics.

- *Knowledge and technologies that aid interpretation of information*: Interpret the confounding states as "disturbance" terms. For example, measurement of a neurochemical in blood samples may correspond to observations of the signal $X = \omega + b + \epsilon$, where the confounding state b has a higher variance if the technology is less developed. The difference between the noise term b and the noise term ϵ is that b is persistent, and so its variance can be reduced over time, while the variance of ϵ is fixed. Separating these two allows us to distinguish between reducible and irreducible noise in the signal.

Decision Environment. A sequence of agents indexed by time t move sequentially. Each agent chooses one of the N sources and observes a realization of the corresponding signal. He then predicts ω, selecting an action $a \in \mathbb{R}$ and receiving the payoff $-(a-\omega)^2$. We assume throughout that all signal realizations are public. Thus, each agent t faces a history $h^{t-1} \in ([N] \times \mathbb{R})^{t-1} = H^{t-1}$ consisting of all past signal choices and their realizations, and his signal acquisition strategy is a function from histories to sources. The agent's optimal prediction of ω is his posterior mean, and his expected payoff is the negative of his posterior variance of ω. At every history h^{t-1}, the agent's expected payoffs are maximized by choosing the signal that minimizes his posterior variance of ω.

Since the environment is Gaussian, posterior variance of ω is a deterministic function $V(q_1, \ldots, q_N)$ of the number of times q_i that each signal i has been observed so far.[8] Thus, each agent's signal acquisition is a function of past signal acquisitions only (and not of the signal realizations). This allows us to track society's acquisitions as deterministic *count vectors*

$$m(t) = (m_1(t), \ldots, m_N(t))' \in \mathbb{Z}_+^N$$

where $m_i(t)$ is the number of times that signal i has been observed up to and including period t. The count vector $m(t)$ evolves according to the following rule: $m(0)$ is the zero

[8] For a normal prior and normal-linear signals, the posterior covariance matrix does not depend on signal realizations. See Appendix A.1 for the complete closed-form expression for V.

vector, and for each time $t \geq 0$ there exists $i^* \in \text{argmin}_i V(m_i(t)+1, m_{-i}(t))$ such that

$$m_i(t+1) = \begin{cases} m_i(t)+1 & \text{if } i = i^* \\ m_i(t) & \text{otherwise} \end{cases}$$

That is, the count vector increases by 1 in the coordinate corresponding to the signal that yields the greatest immediate reduction in posterior variance. We allow ties to be broken arbitrarily, and there may be multiple possible paths $m(t)$.

We are interested in the *long-run frequencies* of observation $\lim_{t\to\infty} m_i(t)/t$ for each source i—that is, the fraction of periods eventually devoted to each source. As we show later in Section 5, these limits exist under a mild technical assumption.

3 Complementary Information

In this section, we introduce a definition for *complementary sets* of sources.

Let $\tau(q_1, \ldots, q_N) = 1/V(q_1, \ldots, q_N)$ be the posterior precision about the payoff-relevant state ω given q_i observations of each source i, with $\tau_0 := \tau(0, \ldots, 0)$ representing the prior precision. We define the *informational value* of a set of sources $\mathcal{S} \subseteq [N] := \{1, \ldots, N\}$ to be the largest improvement on precision, averaged across periods, that agents can achieve by acquiring signals from $\mathcal{S}$ alone.[9]

Definition 1. *The (asymptotic)* informational value *of the set $\mathcal{S}$ is the maximal average increase in the precision about ω over a long horizon:*

$$\text{val}(\mathcal{S}) = \limsup_{t\to\infty,\ q^t \in Q^t_{\mathcal{S}}} \frac{\tau(q^t) - \tau_0}{t}$$

where the limit is along a sequence of $q^t \in Q^t_{\mathcal{S}}$ with $t \to \infty$, and where

$$Q^t_{\mathcal{S}} = \left\{ q \in \mathbb{Z}^N_+ : \textstyle\sum_{i=1}^N q_i = t \text{ and } \text{supp}(q) \subset \mathcal{S} \right\}$$

is the set of all count vectors that allocate t observations across (only) the sources in $\mathcal{S}$.

We emphasize that the informational value is defined with respect to learning about ω. But since we fix the payoff-relevant state throughout this paper, we will omit the dependence of this value on ω. Separately, we highlight that a key property of the above definition is that informational value is *prior-independent* (see Claim 1 below).

The following definition of a *complementary set* is based on Borgers, Hernando-Veciana and Krahmer (2013).

[9] This definition of informational value closely resembles the definition of the value of a team in Chade and Eeckhout (2018), although we consider the precision of beliefs instead of negative posterior variance. Using $\text{val}(\mathcal{S}) = \limsup_{t\to\infty, q^t\in Q^t_{\mathcal{S}}} -tV(q^t)$ in Definition 2 returns a similar notion of complementarity, but presents a technical issue of evaluating $\infty - \infty$ since such a value could be $-\infty$.

Definition 2. *The set $\mathcal{S}$ is* complementary *if* $\text{val}(\mathcal{S}) > \text{val}(\emptyset)$ *and*

$$\text{val}(\mathcal{S}) - \text{val}(\mathcal{S}\backslash\mathcal{S}') > \text{val}(\mathcal{S}') - \text{val}(\emptyset)$$

for all nonempty proper subsets $\mathcal{S}'$ of $\mathcal{S}$.

Informally, a set $\mathcal{S}$ is complementary if the set is informative (the set's informational value exceeds that of the empty set), and the marginal value of having access to the sources in any $\mathcal{S}' \subset \mathcal{S}$ is enhanced by also having access to the sources $\mathcal{S}\backslash\mathcal{S}'$. We note that the first condition that $\text{val}(\mathcal{S}) > \text{val}(\emptyset)$ is implied by our second condition whenever $\mathcal{S}$ is not a singleton.[10]

Rewriting the condition in Definition 2 as

$$\frac{1}{2}\,\text{val}(\mathcal{S}) + \frac{1}{2}\,\text{val}(\emptyset) > \frac{1}{2}\,\text{val}(\mathcal{S}') + \frac{1}{2}\,\text{val}(\mathcal{S}\backslash\mathcal{S}')$$

we can interpret the definition a second way. Suppose the set of available sources is to be determined by a lottery at time $t = 0$, after which the Social Planner acquires information optimally from the (realized) available set of sources, valuing each set $\mathcal{S}$ at $\text{val}(\mathcal{S})$. Compare the choice between: (1) access to all of the sources in $\mathcal{S}$ with probability $1/2$, and otherwise no access to information; and (2) an equal probability of access to the sources in $\mathcal{S}' \subsetneq \mathcal{S}$ and access to the sources $\mathcal{S}\backslash\mathcal{S}'$. If (1) yields a higher value than (2) for all nonempty $\mathcal{S}' \subsetneq \mathcal{S}$, then we say that the set of sources $\mathcal{S}$ is complementary. Thus, the condition above implies that there is extra value to having access to *all* of the sources in $\mathcal{S}$.

3.1 Discussion

Our definition and interpretations above closely mirror the constructions in Borgers, Hernando-Veciana and Krahmer (2013) for complementary pairs of signals, but differ in a few key ways:

First, Definition 2 is for sets of signals, while Borgers, Hernando-Veciana and Krahmer (2013) focuses on pairs. Indeed, "complementary" is often used to describe pairs of objects, e.g., an encoded message is complementary to the key for that code. Our definition extends this idea to sets, where the generalization can be understood in either of two ways. First, we might consider a set to be complementary if *all* of the pieces combine to enhance the whole.[11] For example, the sources

$$X_1 = \omega + b_1 + \epsilon_1$$
$$X_2 = b_1 + b_2 + \epsilon_2$$
$$X_3 = b_2 + \epsilon_3$$

[10] The condition has bite when $|\mathcal{S}| = 1$, and in particular rules out any singleton confounded signal.

[11] According to the Oxford Pocket English Dictionary, *complementary* means "combining in such a way as to enhance or emphasize the qualities of each other or another." For example: "three guitarists playing interlocking, complementary parts."

are complementary, since the presence of each is critical to enhancing the value of the others (ω can only be learned by observing all three sources). Another possibility is to require that each *pair of subsets* that partition the whole set are complementary.[12] In this case, our conceptual extension is not to many complementary sources, but rather to pairs of complementary sets. In the example above, we might say that access to the set $\{X_1, X_2\}$ is complementary to access to $\{X_3\}$, and likewise for the other combinations. Our proposed Definition 2 is stated in terms of the second perspective, but it turns out that complementary sets are also characterized by having an informational value strictly greater than all proper subsets, thus relating to the first perspective that each piece contributes to the whole.[13]

Second, we consider complementary *sources* as opposed to complementary *signal observations*. That is, our definition does not ask whether *a single observation* of some signal improves the marginal value of a single observation of another. Rather, we ask whether *access* to some source improves the marginal value of access to another, where the Social Planner can optimally allocate many observations across the sources to which he has access.[14] However, the two concepts are related, as we show later in Section 6.4.

Finally, Borgers, Hernando-Veciana and Krahmer (2013) considers a notion of complementarity that is *uniform* across all decision problems. In our setting, we focus on decisions that depend only on the payoff-relevant state ω, and we have further used the specific val function as a metric. However, our definition of complementary sets *is* robust to any monotone transformation of informational values.[15] As mentioned, our definition also turns out to be uniform across prior beliefs, which is in line with Borgers, Hernando-Veciana and Krahmer (2013).

[12]Under this interpretation, the name "complementary set" would be a slight abuse of terminology, standing in for the longer statement that every pair of subsets partitioning the whole set are complementary.

[13]Formally, Proposition 1 and part (a) of Proposition 2 below imply that a set $\mathcal{S}$ is complementary according to Definition 2 if and only if $\text{val}(\mathcal{S}) > \text{val}(\mathcal{S}')$ for every proper subset $\mathcal{S}'$ of $\mathcal{S}$, including $\mathcal{S}' = \emptyset$.

[14]The key difficulty with defining complementarity based on the value of one additional signal observation is that these marginal values are prior- and history-dependent. Using such a notion, whether a set of sources constitutes a complementary set would depend on the (endogenous) history of signal acquisitions.

[15]To see this, first suppose $\mathcal{S}$ is complementary under the current definition. Using the characterization in Proposition 1, we have $\text{val}(\mathcal{S}) > \text{val}(\mathcal{S}\backslash\mathcal{S}') = \text{val}(\mathcal{S}') = \text{val}(\emptyset)$, so the inequality $\text{val}(\mathcal{S}) - \text{val}(\mathcal{S}\backslash\mathcal{S}') > \text{val}(\mathcal{S}') - \text{val}(\emptyset)$ is preserved under monotone transformations of the val function. On the other hand, suppose $\mathcal{S}$ is not complementary under the current definition. There are two cases: Either $\mathcal{S}$ contains no complementary subsets, or $\mathcal{S}$ strictly contains a complementary subset. In the former case $\text{val}(\mathcal{S}) = \text{val}(\mathcal{S}\backslash\mathcal{S}') = \text{val}(\mathcal{S}') = \text{val}(\emptyset)$, which is preserved under monotone transformations. In the latter case there exists a complementary set $\mathcal{S}' \subsetneq \mathcal{S}$ such that $\text{val}(\mathcal{S}') = \text{val}(\mathcal{S})$, and clearly $\text{val}(\mathcal{S}\backslash\mathcal{S}') \geq \text{val}(\emptyset)$. So $\text{val}(\mathcal{S}) - \text{val}(\mathcal{S}\backslash\mathcal{S}') \leq \text{val}(\mathcal{S}') - \text{val}(\emptyset)$ is again preserved under transformations.

3.2 Characterization

The following result characterizes complementary sets.

Proposition 1. *$\mathcal{S}$ is a complementary set if and only if the first coordinate vector in $\mathbb{R}^K$ admits a unique decomposition*

$$(1, 0, \ldots, 0)' = \sum_{i \in \mathcal{S}} \beta_i^{\mathcal{S}} \cdot c_i$$

where all coefficients $\beta_i^{\mathcal{S}}$ are nonzero.

Thus, a set $\mathcal{S}$ is complementary if its signals uniquely combine to produce an unbiased signal about ω. This characterization allows us to easily identify complementary sets based on their signal coefficient vectors:

Example 1. The set of signals $\{X_1, X_2, X_3\}$ above is complementary, since $(1, 0, 0)' = c_1 - c_2 + c_3$ (where $c_1 = (1, 1, 0)'$ is the coefficient vector associated with X_1, $c_2 = (0, 1, 1)'$ is the coefficient vector associated with X_2, and $c_3 = (0, 0, 1)'$ is the coefficient vector associated with X_3). In contrast, the set of signals $\{X_4, X_5\}$ with $X_4 = \omega + \epsilon_1$ and $X_5 = 2\omega + \epsilon_2$ is not complementary, since many different linear combinations of c_4 and c_5 produce $(1, 0)$. The set $\{X_1, X_2, X_3, X_4\}$ is also not complementary, although it contains multiple complementary subsets.

The next claim characterizes the informational value of a complementary set.

Claim 1. *Let $\mathcal{S}$ be a complementary set. Then, the value of the set $\mathcal{S}$ is:*

$$val(\mathcal{S}) = \left(\frac{1}{\sum_{i \in \mathcal{S}} |\beta_i^{\mathcal{S}}|} \right)^2$$

where $\beta_i^{\mathcal{S}}$ are the ones given in Proposition 1.

More generally, $\text{val}(\mathcal{S})$ can be determined for an arbitrary set $\mathcal{S}$ as follows: If $\mathcal{S}$ contains at least one complementary subset, then its value is equal to the *highest value among its complementary subsets*; otherwise its value is zero. This will follow from the later Proposition 2 part (a).

Throughout the paper, we assume that there is at least one complementary set, and also that complementary sets can be completely ordered based on their informational values.

Assumption 1. *There is at least one complementary set $\mathcal{S} \subseteq [N]$.*

Assumption 2. *Each complementary set has a distinct informational value; that is,* $\text{val}(\mathcal{S}) \neq \text{val}(\mathcal{S})'$ *for all complementary sets $\mathcal{S} \neq \mathcal{S}'$.*

The first assumption is without loss,[16] while the second assumption is *generically* satisfied. In particular, Assumption 2 implies the existence of a "best" complementary set, whose informational value is largest among complementary sets. This set plays an important role, and we will call it $\mathcal{S}^*$ in the remainder of this paper.

4 Optimal Long-Run Observations

In this section, we show that optimal information acquisitions eventually concentrate on the best complementary set $\mathcal{S}^*$. Specifically, consider the distribution

$$\lambda_i^* = \begin{cases} \frac{|\beta_i^{\mathcal{S}^*}|}{\sum_{j \in \mathcal{S}^*} |\beta_j^{\mathcal{S}^*}|} & \forall\ i \in \mathcal{S}^* \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

which assigns zero frequency to signals outside of the best set $\mathcal{S}^*$, and samples sources within $\mathcal{S}^*$ proportionally to the magnitude of $\beta_i^{\mathcal{S}^*}$. That is, each signal in $\mathcal{S}^*$ receives frequency proportional to its contribution to an unbiased signal about ω, as defined in Proposition 1. The result below shows two senses in which λ^* is the optimal long-run frequency over signals.

Proposition 2. *(a)* Optimal Information Aggregation:

$$\text{val}([N]) = \text{val}(\mathcal{S}^*).$$

Additionally, for any sequence $q(t)$ such that $\lim_{t\to\infty} \frac{\tau(q^t)-\tau_0}{t} = \text{val}([N])$, it must hold that $\lim_{t\to\infty} \frac{q(t)}{t} = \lambda^*$.

(b) Social Planner Problem: *For any δ, let $d_\delta(t)$ be the vector of signal counts (up to period t) associated with any strategy that maximizes the δ-discounted average payoff*

$$U_\delta := -\mathbb{E}\left[(1-\delta)\sum_{t=1}^{\infty} \delta^{t-1} \cdot (a_t - \omega)^2\right]$$

Then there exists $\underline{\delta} < 1$ such that for any $\delta \geq \underline{\delta}$ the following holds:

$$\lim_{t\to\infty} \frac{d_\delta(t)}{t} = \lambda^*.$$

[16] Our results extend to situations where ω *cannot* be identified from the available signals. To see this, we first take a linear transformation and work with the following equivalent model: The state vector $\tilde{\theta}$ is K-dimensional *standard Gaussian*, each signal $X_i = \langle \tilde{c}_i, \tilde{\theta}\rangle + \epsilon_i$, and the payoff-relevant parameter is $\langle u, \tilde{\theta}\rangle$ for some constant vector u. Let R be the subspace of $\mathbb{R}^K$ spanned by $\tilde{c}_1, \dots, \tilde{c}_N$. Then project u onto R: $u = r + w$ with $r \in R$ and w orthogonal to R. Thus $\langle u, \tilde{\theta}\rangle = \langle r, \tilde{\theta}\rangle + \langle w, \tilde{\theta}\rangle$. By assumption, the random variable $\langle w, \tilde{\theta}\rangle$ is independent from any random variable $\langle c, \tilde{\theta}\rangle$ with $c \in R$ (because they have zero covariance). Thus the uncertainty about $\langle w, \tilde{\theta}\rangle$ cannot be reduced upon any signal observation. Consequently, agents only seek to learn about $\langle r, \tilde{\theta}\rangle$, returning to the case where the payoff-relevant parameter *is* identified.

Part (a) says that the informational value of $\mathcal{S}^*$ is the same as the informational value of the entire set of available signals. In this sense, having access to all available sources does not improve upon the speed of learning achievable from the best complementary set $\mathcal{S}^*$ alone. Moreover, this speed of learning is attainable *only if* the long-run frequency over sources is the distribution λ^*.[17] Part (b) of Proposition 2 says that a (patient) social planner—who maximizes a discounted average of agent payoffs—will eventually observe sources in the proportions described by λ^*.

Based on these results, we subsequently use λ^* as the optimal benchmark against which to compare society's long-run information acquisitions.

5 Main Results

We now ask whether society's acquisitions converge to the optimal long-run frequencies λ^* characterized above. We show that informational environments can be classified into two kinds—those for which efficient information aggregation is guaranteed (long-run frequencies are λ^*), and those for which "learning traps" are possible (agents exclusively observe some set of sources different from the best set $\mathcal{S}^*$). Separation of these two classes depends critically on the size of the smallest complementary set.

5.1 Learning Traps vs. Efficiency

The following example demonstrates that efficient information aggregation need not occur. Indeed, the set of signals that are observed in the long run can be disjoint from the optimal set $\mathcal{S}^*$.

Example 2. There are three available signals:

$$X_1 = \omega + \epsilon_1$$
$$X_2 = 3\omega + b_1 + \epsilon_2$$
$$X_3 = b_1 + \epsilon_3$$

Both $\{X_1\}$ and $\{X_2, X_3\}$ are complementary sets, but optimal information acquisitions (as defined in Section 4) should eventually concentrate on $\{X_2, X_3\}$.[18]

[17] This result builds on Chaloner (1984), which shows that a "c-optimal experiment design" exists on at most K points. Part (a) additionally supplies a characterization of the optimal design itself and demonstrates uniqueness. One technical difference between our work and Chaloner (1984) is that she studies the optimal continuous design, while we impose an integer constraint on signal counts.

[18] It is straightforward to verify that $\text{val}(\{X_1\}) = 1 < 9/4 = \text{val}(\{X_2, X_3\})$. Note also that $X_2 - X_3$ is an unbiased signal about ω, and it is more informative than two realizations of X_1; this demonstrates $\{X_2, X_3\}$ is the better complementary set without direct computation of informational values.

Now suppose that agents' prior beliefs are such that ω and b_1 are independent, and the prior variance of b_1 is large (exceeds 8). In the first period, observation of X_1 is most informative about ω, since X_2 is perceived as a noisier signal about ω than X_1, and X_3 provides information only about the confounding term b_1 (which is uncorrelated with ω). Agent 1's acquisition of X_1 does not update the variance of b_1, so the same argument shows that agent 2 acquires X_1. Iterating, we have that *every* agent observes signal X_1. In this way, the set $\{X_1\}$ represents a learning trap.[19]

Returning to researchers who sequentially acquire information to learn about the impact of sleep loss on depression, we can interpret the source X_3 as development of a new technology towards this goal—for example, development of a precise, non-invasive tool for measuring levels of the neurotransmitter dopamine in the brain.[20] The source X_2 produces measurements of dopamine using this new method. Repeated development and use of the new method yields larger returns to knowledge (in the long run), and socially it is optimal for researchers to invest towards this path. But if development of the method is slow, each researcher may choose instead to exploit existing technologies for measurement (observation of X_1), maximizing the marginal value of their work but reinforcing the learning trap.

Generalizing Example 2, the result below (stated as a corollary, since it will follow from the subsequent Theorem 1) gives a sufficient condition for learning traps. We impose the following generic assumption on the signal structure, which requires that every set of K signals are linearly independent:

Assumption 3 (Strong Linear Independence)**.** *$N \geq K$ and every K signal coefficient vectors $c_{i_1}, c_{i_2}, \ldots, c_{i_K}$ are linearly independent.*

Corollary 1. *Assume Strong Linear Independence. Then for every complementary set $\mathcal{S}$ with $|\mathcal{S}| < K$, there exists an open set of prior beliefs given which agents exclusively observe signals from $\mathcal{S}$.*

Thus, every small complementary set (fewer than K signals) is a candidate learning trap.

Now suppose in contrast that the smallest complementary set is of size K.[21] Our next result shows that a very different long-run outcome obtains: Starting from *any* prior, society's information acquisition eventually approximates the optimal frequency. Thus, even though

[19]The existence of learning traps is not special to the assumption of normality. We report a related example with non-normal signals in Appendix B.2.

[20]Neurotransmitters are difficult to measure—leading approaches are invasive, requiring insertion of an electrode into the brain, which can only be kept there for a short amount of time, and are not guaranteed to end up next to measurable dopamine. New imaging tools may allow researchers to circumvent these procedures and estimate neurotransmitter levels using non-invasive procedures. See e.g. Badgaiyan (2014)

[21]It follows from Proposition 1 that there are no complementary sets with more than K sources.

agents are short-lived ("myopic"), they end up acquiring information in a way that is socially best.

Corollary 2. *If there are no complementary sets with fewer than K sources, then starting from any prior belief,* $\lim_{t\to\infty} \frac{m_i(t)}{t} = \lambda_i^*$ *holds for every signal i. Thus, efficient information aggregation is guaranteed.*

This result, like Corollary 1 above, follows from the subsequent Theorem 1.

We provide a brief intuition for these results, and in particular for the importance of the number K. Recall that each agent chooses the signal with the highest marginal value (in terms of reducing posterior variance of ω). Thus, if signal acquisitions eventually concentrate on a set $\mathcal{S}$, the marginal values of signals in that set must be persistently higher than marginal values of other signals.

Some, but not all, complementary sets have this property: Observe that each source may belong to *multiple* complementary sets, and the sources within a given complementary set $\mathcal{S}$ can have even stronger complementaries with some other sources. If that were the case, the signal with the highest marginal value might be outside of the set $\mathcal{S}$, making it impossible for society's acquisitions to concentrate on $\mathcal{S}$. Indeed, this logic allows us to show that a complementary set of K sources *cannot* be self-reinforcing unless it is the best set: As observations accumulate from such a set, agents would eventually learn about all of the confounding terms and come to evaluate all sources according to an "objective" asymptotic value. They would perceive sources in the best set $\mathcal{S}^*$ to have higher marginal values, and turn to these, achieving efficient information aggregation as predicted by Corollary 2.

In contrast, if agents observe only $k < K$ sources, then they can have persistent uncertainty about some confounding terms. This may cause society to persistently undervalue those sources confounded by these terms and continually observe signals from a small complementary set. We saw this already in Example 2 where agents failed to obtain any information about the confounding term b_1, and thus persistently undervalued the sources X_2 and X_3. The same intuition applies to Corollary 1.

One may argue that the condition that no complementary set has fewer than K sources is generically satisfied.[22] However, if we expect that sources are endogenous to design or strategic motivations, the relevant informational environments may not fall under this condition. For example, the existence of an unbiased signal about ω (that is, $X = c\omega + \epsilon$) is non-generic in the probabilistic sense, but plausible in practice. Signals that partition into different groups with group-specific confounding terms (as studied in Sethi and Yildiz

[22] We point out that this condition is "generic" in a weaker sense than Assumption 2. To see this, fix the *directions* of coefficient vectors, and suppose that the *precisions* are drawn at random; then, generically different complementary sets have different informational values. In contrast, whether every complementary set has size K is a condition on the *directions* themselves.

(2017)) are also economically interesting but non-generic. Corollary 1 shows that inefficiency is a possible outcome in these cases.

5.2 General Characterization of Long-run Outcomes

Fixing an arbitrary signal structure (which may not satisfy Strong Linear Independence), we now provide a complete characterization of the possible long-run observation sets as the prior belief varies. We introduce a new definition, which strengthens the notion of a complementary set:

Definition 3. *$\mathcal{S}$ is a* strongly complementary *set if it is complementary, and* $\text{val}(\mathcal{S}) > \text{val}(\mathcal{S}')$ *for all sets $\mathcal{S}'$ that differ from $\mathcal{S}$ in exactly one source ($|\mathcal{S} - \mathcal{S}'| = |\mathcal{S}' - \mathcal{S}| = 1$).*[23]

The property of *strongly complementary* can be understood as requiring that the set is complementary and also something more: These complementarities are "locally best" in the sense that it is not possible to obtain stronger complementarities by swapping out just one source. We point out that while the definition of complementary sets does not depend on the ambient set (i.e., $[N]$) of available sources, the notion of strongly complementary does.

Example 3. Suppose the available signals are $X_1 = \omega + b_1 + \epsilon_1$, $X_2 = b_1 + \epsilon_2$, and $X_3 = 2b_1 + \epsilon_3$. Then the set $\{X_1, X_2\}$ is complementary but not strongly complementary, since $\text{val}(\{X_1, X_3\}) > \text{val}(\{X_1, X_2\})$.

Our main result generalizes both the learning traps result and the efficient information aggregation result from the previous section. Theorem 1 says that long-run information acquisitions eventually concentrate on a set $\mathcal{S}$ (starting from some prior belief) *if and only if* $\mathcal{S}$ is strongly complementary.

Theorem 1. *The set $\mathcal{S}$ is strongly complementary $\Longleftrightarrow$ there exists an open set of prior beliefs given which agents eventually exclusively observe signals from $\mathcal{S}$ (that is, long-run frequencies exist and have support in $\mathcal{S}$).*

When there is a single strongly complementary set, then all priors must lead to this set. Our previous Corollary 2 provides a sufficient condition that implies this, and moreover gives that the unique strongly complementary set is the *best* complementary set. When there are multiple strongly complementary sets, then different priors lead to different long-run outcomes, some of which are inefficient. Our previous Corollary 1 describes a sufficient condition for such multiplicity.

[23] Indeed, the requirement that $\mathcal{S}$ is complementary is extraneous: One can show using Proposition 1 that if $\text{val}(\mathcal{S}) > \text{val}(\mathcal{S}')$ for all sets $\mathcal{S}'$ differing from $\mathcal{S}$ in exactly one source, then $\mathcal{S}$ must be complementary.

6 Proof Outline for Theorem 1

6.1 Asymptotic Variance V^*

We first introduce the following *normalized asymptotic* posterior variance function V^*, which takes frequency vectors $\lambda \in \Delta^{N-1}$ as input:

$$V^*(\lambda) = \lim_{t\to\infty} t \cdot V(\lambda t).$$

This function is convex in λ and its unique minimum is the optimal frequency vector λ^* (Lemma 7 in the appendix). We also show that at late periods t, the signal choice that minimizes V also *approximately* minimizes V^* (Lemma 14).

For simplicity of explanation, we will assume throughout this section that at large t, the signal choice that minimizes V *exactly* minimizes V^*. Then, the frequency vector $\lambda(t) := \frac{m(t)}{t}$ evolves in the coordinate direction that minimizes V^*. We will refer to this as *coordinate descent*. Unlike the usual gradient descent, coordinate descent is restricted to move in coordinate directions. This restriction corresponds to our assumption that each agent can only acquire a discrete signal (rather than a mixture of signals).

One case where coordinate descent coincides with gradient descent is when V^* is everywhere differentiable: Differentiability ensures that all directional derivatives can be written as convex combinations of partial derivatives along coordinate directions. In that case, evolution of $\lambda(t)$ would necessarily end at the global minimizer λ^*, implying efficient information aggregation.

6.2 Differentiability of V^*

The function V^*, however, is *not* guaranteed to be differentiable everywhere. Consider our learning trap example with signals $X_1 = \omega + \epsilon_1$, $X_2 = 3\omega + b_1 + \epsilon_2$, and $X_3 = b_1 + \epsilon_3$. It can be computed that the asymptotic variance function is

$$V^*(\lambda_1, \lambda_2, \lambda_3) = \lambda_1 + \frac{9}{1/\lambda_2 + 1/\lambda_3}.$$

Consider the frequency vector $\lambda = (1, 0, 0)$. From the above formula it is easy to verify that the asymptotic variance $V^*(\lambda)$ is increased if we perturb λ by re-assigning weight from X_1 to X_2, or from X_1 to X_3. But V^* is reduced if we re-assign weight from X_1 to *both* X_2 and X_3, evenly. This means that the derivative of V^* in either direction $(-1, 1, 0)$ or $(-1, 0, 1)$ is positive, while its derivative in the direction $(-1, \frac{1}{2}, \frac{1}{2})$ is in fact negative. So V^* is not differentiable at λ.

Coordinate descent can become stuck at vectors λ such as this, so that agents repeatedly sustain the frequency vector λ instead of moving (in a non-coordinate direction) to a different frequency vector with smaller V^*. This is exactly what creates learning traps.

A sufficient condition for V^* to be differentiable at some frequency vector turns out to be that the signals receiving positive frequencies span all of $\mathbb{R}^K$.[24] This explains the result in Corollary 2: When each complementary set consists of K signals, society *has to* observe at least K signals in order to learn the payoff-relevant state ω. Thus, in the process of learning about ω, agents necessarily observe a set of signals that span $\mathbb{R}^K$, leading to efficient information aggregation.

6.3 Generalization to Arbitrary Subspaces

Now observe that our arguments above were not special to considering the whole space $\mathbb{R}^K$. If we restrict the available sources to some subset of $[N]$, and look at the subspace of $\mathbb{R}^K$ spanned by these sources, then our previous analysis will apply to this restricted space.

Specifically, given any prior belief, define $\mathcal{S}$ to be the set of sources that agents eventually observe. Let $\overline{\mathcal{S}}$ be the available signals that can be reproduced as a linear combination of signals from $\mathcal{S}$. In other words, these sources belong to the "subspace spanned by $\mathcal{S}$." We can consider the restriction of V^* to all frequency vectors with support in $\overline{\mathcal{S}}$. Parallel to the discussion above, the restricted version of V^* is both convex and differentiable in this subspace (at frequency vectors that assign positive weights to signals in $\mathcal{S}$). Thus, coordinate descent must lead to the minimizer of V^* in this subspace.

Just as the overall optimal frequency vector λ^* is supported on the best complementary set $\mathcal{S}^*$, the "locally optimal" frequency vector that minimizes V^* in the subspace is supported on the best complementary set within $\overline{\mathcal{S}}$. So our assumption that agents eventually concentrate signal acquisitions on the set $\mathcal{S}$ is valid only if $\mathcal{S}$ is *best in its subspace*; that is, $\text{val}(\mathcal{S}) = \text{val}(\overline{\mathcal{S}})$.

6.4 An Equivalence Result

The lemma below relates the property of "best in its subspace" to the notion of "strongly complementary."

Lemma 1. *The following conditions are equivalent for a complementary set $\mathcal{S}$:*

(a) $\text{val}(\mathcal{S}) = \text{val}(\overline{\mathcal{S}})$.

(b) $\mathcal{S}$ is strongly complementary.

[24]In the example above, V^* is differentiable at λ whenever λ has two strictly positive coordinates.

(c) *For any* $i \in \mathcal{S}$ *and* $j \notin \mathcal{S}$*,* $\partial_i V^*(\lambda^{\mathcal{S}}) < \partial_j V^*(\lambda^{\mathcal{S}})$*, where* $\lambda^{\mathcal{S}}$ *(proportional to* $|\beta^{\mathcal{S}}|$*) is the optimal frequency vector supported on* $\mathcal{S}$*.*

This lemma states that a strongly complementary set $\mathcal{S}$ is "locally best" in three different senses. Part (a) says such a set has the highest informational value in its subspace. Part (b) says its informational value is higher than any set obtained by swapping out one source. Part (c) says that starting from the optimal sampling rule over $\mathcal{S}$, re-allocating frequencies from signals in $\mathcal{S}$ to any other signal increases the posterior variance and reduces speed of learning.

The implication from part (a) to part (b) is straightforward: Suppose $\mathcal{S}$ is best in its subspace, and $\mathcal{S}'$ is obtained from $\mathcal{S}$ by removing signal i and adding signal j. Then the informational value of $\mathcal{S}'$ is either zero, or equal to the value of a complementary subset that necessarily includes signal j. In the latter case j must belong to the subspace spanned by $\mathcal{S}$, implying that $\mathcal{S}' \subset \overline{\mathcal{S}}$. So $\text{val}(\mathcal{S}') \leq \text{val}(\overline{\mathcal{S}}) = \text{val}(\mathcal{S})$, and inequality must be strict because complementary sets have different values.

We next show part (b) implies part (c). Suppose part (c) fails, so some perturbation moving weight from source $i \in \mathcal{S}$ to source $j \notin \mathcal{S}$ would decrease V^*. Then, by definition of informational value, we would have $\text{val}(\mathcal{S} \cup \{j\}) > \text{val}(\mathcal{S})$. But as Proposition 2 part (a) suggests, the value of $\mathcal{S} \cup \{j\}$ is equal to the highest value among its complementary subsets. Strong complementarity of $\mathcal{S}$ ensures that $\mathcal{S}$ is the best complementary subset of $\mathcal{S} \cup \{j\}$. Thus we obtain $\text{val}(\mathcal{S} \cup \{j\}) = \text{val}(\mathcal{S})$, leading to a contradiction.

Finally, part (c) implies that $\lambda^{\mathcal{S}}$ is a local minimizer of V^* in the subspace spanned by $\mathcal{S}$ (where the restriction of V^* is differentiable). Since V^* is convex, the frequency vector $\lambda^{\mathcal{S}}$ must in fact be a "global" minimizer of V^* in this subspace. Hence $\mathcal{S}$ is best in its subspace and part (a) holds.

6.5 Completing the Argument

The arguments above tell us that information acquisitions eventually concentrate on a strongly complementary set, delivering one direction of Theorem 1: $\mathcal{S}$ is a long-run outcome only if $\mathcal{S}$ is strongly complementary.

To prove the "if" direction, we directly construct priors such that a given strongly complementary set $\mathcal{S}$ is the long-run outcome. The construction generalizes the idea in Example 2, where we assign high uncertainty to those confounding terms that do not afflict signals in $\mathcal{S}$ (as well as those in the same subspace $\overline{\mathcal{S}}$), and low uncertainty to those that do. This asymmetry guarantees that signals from $\overline{\mathcal{S}}$ have persistently higher marginal values than the remaining signals. Lastly, we use part (c) of the above Lemma 1 to show that agents focus on observing from $\mathcal{S}$, rather than the potentially larger set $\overline{\mathcal{S}}$. Indeed, if the historical frequency of acquisitions is close to $\lambda^{\mathcal{S}}$, then signals in $\mathcal{S}$ have higher marginal values than the remaining signals in their subspace; and as these signals in $\mathcal{S}$ continue to be chosen, society's frequency vector remains close to $\lambda^{\mathcal{S}}$. This completes the proof of Theorem 1.

7 Interventions

The previous sections demonstrate the possibility for agents to persistently acquire suboptimal sources of information. This naturally suggests a question of what kinds of policies might free agents from these learning traps. We compare several possible policy interventions: Increasing the *quality* of information acquisition (so that each signal acquisition is more informative); restructuring incentives so that agents' payoffs are based on information obtained over several periods (equivalent to acquisition of *multiple signals* each period); and providing a one-shot release of *free information*, which can then guide subsequent acquisitions.

7.1 More Precise Information

Consider first an intervention in which the precision of each signal draw is uniformly increased. For example, if different signals correspond to measurement of different neurochemicals in a group of lab subjects, a government agency can provide researchers with funding that permits recruitment of more subjects. This improves the quality of the estimate regardless of which neurochemical the researcher chooses to measure.

We model this intervention by supposing that each signal acquisition now produces B independent observations from that source (with the main model corresponding to $B = 1$). The result below shows that providing more informative signals is of limited effectiveness: Any set of signals that is a potential learning trap given $B = 1$ remains a potential learning trap under arbitrary improvements to signal precision.

Corollary 3. *Suppose that for $B = 1$, there is a set of priors given which signals in $\mathcal{S}$ are exclusively viewed in the long run. Then, for every $B \in \mathbb{Z}_+$, there is a set of priors given which these signals are exclusively viewed in the long run.*

This corollary follows directly from Theorem 1.

However, the *set of* prior beliefs that yield $\mathcal{S}$ as a long-run outcome need not be the same as B varies. For a *fixed* prior belief, subsidizing higher quality acquisitions may or may not move the community out of a learning trap. To see this, consider first the signal structure and prior belief from Example 2. Increasing the precision of signals is ineffective there: As long as the prior variance on b is larger than 8, each agent still chooses signal X_1 regardless of signal precision. In Appendix B.3, we provide a contrasting example in which increasing the precision of signals can indeed break agents out of a learning trap from a specified prior belief.

7.2 Batches of Signals

Another possibility is to restructure the incentive scheme so that agents' payoffs are based on information acquired from multiple signals. In practice, this might mean that payoffs are determined after a given time interval: For example, researchers may be evaluated based on a set of papers, so that they want to maximize the impact of the entire set. Alternatively, agents might be given the means to acquire multiple signals each period: For example, researchers may be arranged in labs, with a principal investigator directing the work of multiple individuals simultaneously.

Formally, we suppose here that each agent can allocate B observations across the sources (where $B = 1$ returns the main model). Note the key difference from the previous intervention: It is now possible for the B observations to be allocated across *different* signals. This difference enables agents to take advantage of the presence of complementarities, and we show that efficient information aggregation can be guaranteed in this case:

Proposition 3. *For sufficiently B, if each agent acquires B signals every period, then long-run frequency is λ^* starting from every prior belief.*

Thus, given sufficiently many observations each period, agents will allocate observations in a way that approximates the optimal frequency.

The number of observations needed, however, depends on details of the informational environment. In particular, the required B cannot be bounded as a function of the number of states K and number of signals N.[25] See Appendix A.7 for further details.

7.3 Free Information

Finally, we consider provision of free information to the community. We can think of this as releasing information that a policymaker knows, or as a reduced form for funding specific kinds of research, the results of which are then made public.

[25]The required B depends on two properties: first, on how well the optimal frequency λ^* can be approximated via allocation of B observations—for example, $\lambda^* = (1/2, 1/2)$ can be achieved exactly using two observations, while $\lambda^* = (3/8, 5/8)$ cannot; second, on the difference in learning speed between the best set and the next best complementary set, which determines the "slack" that is permitted in the approximation of λ^*. Thus, a small batch size B is sufficient when the optimal frequency λ^* can be well-approximated using a small number of observations, or when there are large efficiency gains from observing the best set.

Formally, the policymaker chooses several signals $X_j = \langle p_j, \theta \rangle + \mathcal{N}(0,1)$, where each $\|p_j\|_2 \leq \gamma$, so that signal precisions are bounded by γ^2. At time $t = 0$, independent realizations of these signals are made public. All subsequent agents update their prior beliefs based on this free information in addition to the history of signal acquisitions thus far.

We show that given a sufficient number of (different kinds of) signals, efficient learning can be guaranteed. Specifically, if $k \leq K$ is the size of the optimal set $\mathcal{S}^*$, then $k-1$ precise signals are sufficient to guarantee efficient learning:

Proposition 4. *Let $k := |\mathcal{S}^*|$. Under Unique Minimizer, there exists a $\gamma < \infty$, and $k-1$ signals $X_j = \langle p_j, \theta \rangle + \mathcal{N}(0,1)$ with $\|p_j\|_2 \leq \gamma$, such that with these free signals provided at $t = 0$, society's long-run frequency is λ^* starting from every prior belief.*

The proof is by construction. We show that as long as agents understand those confounding terms that appear in the best set of signals (these parameters have dimension $k-1$), they will come to discover this best set.[26]

This intervention is most relevant in settings in which a technological advance could greatly speed up progress, but development of the technology is slow and tedious, such as described in Section 5.1. The government can intervene by funding preliminary development of the new technology, which then encourages researchers to begin using it. Once use of the technology is common, the payoff to advancing the technology increases, and even myopic researchers may contribute to this agenda. In this way, provision of free information can nudge agents onto the right path of learning.

8 Welfare Loss Under Learning Traps

We conclude with a more detailed analysis of the size of welfare losses under learning traps, and a generalization of our model in which the unknown states evolve over time.

8.1 Welfare Criteria

We focus on two classic welfare criteria: the speed at which information is aggregated (see e.g. Vives (1992), Golub and Jackson (2012), and Harel et al. (2018)) and discounted average payoffs (see e.g. Easley and Kiefer (1988) and Aghion et al. (1991)).

[26] This intervention requires knowledge of the full correlation structure, and also which set $\mathcal{S}^*$ is best. An alternative intervention, with higher demands on information provision but lower demands on knowledge of the environment, is to provide $K-1$ (sufficiently precise) signals about all of the confounding terms.

Information Aggregation. A small modification of Example 2 shows that society's long-run speed of learning can be arbitrarily slower than the optimal speed. Specifically, the informational value of the best complementary set can be arbitrarily large relative to the value of the set that agents eventually observe.

Example 4. There are three available signals:

$$X_1 = \frac{1}{L}\omega + \epsilon_1$$
$$X_2 = \omega + b_1 + \epsilon_2$$
$$X_3 = b_1 + \epsilon_3$$

where $L > 0$ is a constant. In this example, the ratio

$$\operatorname{val}(\{X_2, X_3\}) / \operatorname{val}(\{X_1\}) = L^2/4,$$

which increases without bound as $L \to \infty$. But for every choice of L, there is a set of priors given which X_1 is exclusively observed.[27]

Discounted Average Payoffs. Define

$$U_\delta^M = \mathbb{E}_M \left[-\sum_{t=1}^{\infty} (1-\delta)\delta^{t-1}(a_t - \omega)^2 \right]$$

to be the δ-discounted average payoff across agents, who follow a "myopic" signal acquisition strategy with optimal predictions a_t. Also define U_δ^{SP} to be the maximum δ-discounted average payoff, where the Social Planner can use any signal acquisition strategy. Note that both payoff sums are negative, since flow payoffs are quadratic loss at every period.

Again from Example 4, we see that for every constant $c > 0$, there is a signal structure and prior such that the limiting payoff ratio satisfies[28]

$$\lim_{\delta \to 1} U_\delta^M / U_\delta^{SP} > c.$$

Thus, the *payoff ratio* can be arbitrarily large. Note that because payoffs are negative, larger values of the ratio $U_\delta^M / U_\delta^{SP}$ correspond to greater payoff inefficiencies.

[27] The region of inefficient priors (that result in suboptimal learning) does decrease in size as the level of inefficiency increases. Specifically, as L increases, the prior variance of b_1 has to increase correspondingly in order for the first agent to choose X_1.

[28] Example 4 implies the ratio of flow payoffs at late periods can be made arbitrarily large. As $\delta \to 1$, these later payoffs dominate the total payoffs from the initial periods (since the harmonic series diverges). So the ratio of aggregate discounted payoffs is also large.

On the other hand, the *payoff difference* vanishes in the patient limit; that is,

$$\lim_{\delta \to 1} \left(U_\delta^{SP} - U_\delta^{M} \right) = 0$$

in all environments. To see this, note that agents eventually learn ω even while in a learning trap, albeit slowly. Thus flow payoffs converge to zero at large periods, implying $\lim_{\delta \to 1} U_\delta^{SP} = \lim_{\delta \to 1} U_\delta^{M} = 0$.

In what follows, we show this conclusion critically depends on the assumption that unknown states are perfectly persistent. We outline a sequence of autocorrelated models that converge to our main model (with perfect state persistence). At near perfect persistence, welfare losses under learning traps can be arbitrarily large according to both of the above measures.

8.2 Extension: Autocorrelated Model

In our main model, the state vector $\theta = (\omega, b_1, \ldots, b_{K-1})'$ is persistent across time. We now consider a state vector θ^t that evolves according to the following law:

$$\theta^1 \sim \mathcal{N}(0, \Sigma^0); \quad \theta^{t+1} = \sqrt{\alpha} \cdot \theta^t + \sqrt{1-\alpha} \cdot \eta^t, \quad \text{where } \eta^t \sim \mathcal{N}(0, M).$$

Above, means are normalized to zero, and the prior covariance matrix of the state vector at time $t = 1$ is Σ^0. We restrict the autocorrelation coefficient $\sqrt{\alpha}$ to belong to $(0, 1)$. Choice of $\alpha = 1$ returns our main model, and we will be interested in approximations where α is close to but strictly less than 1. The *innovation* $\eta^t \sim \mathcal{N}(0, M)$ captures the additional noise terms that emerge under state evolution, which we assume to be i.i.d. across time. Fixing signal coefficients $\{c_i\}$, every autocorrelated model is indexed by the triple (M, Σ^0, α).

In each period, the available signals are

$$X_i^t = \langle c_i, \theta^t \rangle + \epsilon_i^t, \quad \epsilon_i^t \sim \mathcal{N}(0, 1).$$

The signal noises are i.i.d. across time and further independent from the innovations in state evolution. The agent in period t chooses the signal that minimizes posterior variance of ω^t, while the Social Planner seeks to minimize a discounted sum of such posterior variances.

Theorem 2. *Suppose $\mathcal{S}$ is strongly complementary. Then there exists M, Σ^0 such that for every $\epsilon > 0$, there is an $\underline{\alpha}(\epsilon) < 1$ such that for each autocorrelated model (M, Σ^0, α) with $\alpha > \underline{\alpha}(\epsilon)$:*

1. *Every agent in the autocorrelated model observes a signal from $\mathcal{S}$.*

2. *The resulting discounted average payoff satisfies*

$$\limsup_{\delta\to 1} U_\delta^M \leq -(1-\epsilon)\cdot\sqrt{(1-\alpha)\left(\frac{M_{11}}{\mathrm{val}(\mathcal{S})}\right)}\ ,$$

while it is feasible to achieve a patient payoff of

$$\liminf_{\delta\to 1} U_\delta^{SP} \geq -(1+\epsilon)\cdot\sqrt{(1-\alpha)\left(\frac{M_{11}}{\mathrm{val}(\mathcal{S}^*)}\right)}$$

by sampling from $\mathcal{S}^*$.

Part (1) generalizes Theorem 1, showing that every strongly complementary set is a potential long-run observation set given imperfect persistence. This suggests that the notion of strong complementarity and its important extends beyond our main model.

Part (2) shows that whenever $\mathcal{S}$ is different from the best complementary set $\mathcal{S}^*$, then social acquisitions result in significant payoff inefficiency as measured by the payoff ratio. Indeed, for α close to 1 the ratio $\lim_{\delta\to 1} U_\delta^M/U_\delta^{SP}$ is at least $\sqrt{\mathrm{val}(\mathcal{S}^*)/\,\mathrm{val}(\mathcal{S})}$, which can be arbitrarily large depending on the signal structure.

The following proposition strengthens this statement, using Example 4 to show that the payoff difference between optimal and social acquisitions can also be arbitrarily large:

Proposition 5. *For every $\epsilon > 0$, there exists a signal structure as in Example 4 and an autocorrelated model (M, Σ^0, α) such that $\liminf_{\delta\to 1} U_\delta^{SP} \geq -\epsilon$ but $\limsup_{\delta\to 1} U_\delta^M \leq -\frac{1}{\epsilon}$.*

From this analysis, we take away that learning traps in general result in average payoff losses (and potentially large losses) so long as unknown states are not perfectly persistent over time.

9 Other Extensions

General Payoff Functions. Our main results extend when each agent t chooses an action to maximize an arbitrary individual payoff function $u_t(a_t, \omega)$ (recall that previously we restricted to $u_t(a_t, \omega) = -(a_t - \omega)^2$). We require only that these payoff functions are nontrivial in the following sense:

Assumption 4 (Payoff Sensitivity to Mean)**.** *For every t, any variance $\sigma^2 > 0$ and any action $a^* \in A$, there exists a positive Lebesgue measure of μ for which a^* does* not *maximize $\mathbb{E}[u_t(a, \omega) \mid \omega \sim \mathcal{N}(\mu, \sigma^2)]$.*

That is, for every belief variance, the expected value of ω affects the optimal action to take. This rules out cases with a "dominant" action and ensures that each agent *strictly* prefers to choose the most informative signal. Since the signal that minimizes the posterior variance about ω Blackwell-dominates every other signal (Hansen and Torgersen, 1974), each agent's signal acquisition remains unchanged.

However, the interpretation of the optimal benchmark (that we defined in Section 4) is more limited. Specifically, while the optimal frequency can still be interpreted as maximizing information revelation, the relationship to the social planner problem (part (b) of Proposition 2) may fail. A detailed discussion is relegated to Appendix B.4.1.

Low Altruism. So far we have assumed that agents care only to maximize the accuracy of their own prediction of the payoff-relevant state. Consider a generalization in which agents are slightly altruistic; that is, each agent t chooses a signal as well as an action a_t to maximize discounted payoffs $\mathbb{E}\left[\sum_{t' \geq t} \rho^{t'-t} \cdot (a_t - \omega)^2\right]$, assuming that future agents will behave similarly. (Note that $\rho = 0$ returns our main model.) We show in Appendix B.4.2 that for ρ sufficiently small, part (a) of Theorem 1 continues to hold in every equilibrium of this game. So the existence of learning traps is robust to a small degree of altruism. By Proposition 7 in the appendix, Part (b) of Theorem 1 also extends, showing that strongly complementary sets are the only possible long-run outcomes starting from any prior.

Multiple Payoff-Relevant States. In our main model, only one of the K persistent states is payoff-relevant. Consider a generalization in which each agent predicts (the same) $r \leq K$ unknown states and his payoff is determined via a weighted sum of quadratic losses. We show in Appendix B.4.3 that our main results extend to this setting. The possibility for agents to have payoffs that depend on *heterogeneous* states is also interesting, and we leave this for future work.

10 Conclusion

We study a model of sequential learning, where short-lived agents choose what kind of information to acquire from a large set of available information sources. Because agents do not internalize the impact of their information acquisitions on later decision-makers, they may acquire information inefficiently (from a social perspective). Inefficiency is not guaranteed, however: Depending on the informational environment, myopic concerns can endogenously push agents to identify and observe only the most informative sources.

Our main results separate these possibilities, and reveal that the extent of *learning spillovers* is essential to determining which outcome emerges. Specifically, does information about unknowns of immediate societal interest (i.e., the payoff-relevant state) also teach about unknowns that are only of indirect value (i.e., the confounding terms)?

When such spillovers are present, simple incentive schemes for information acquisition—in which agents care only about immediate contributions to knowledge—are sufficient for efficient long-run learning. When these spillovers are not built into the environment, other incentives are needed. For example, forward-looking funding agencies can encourage investment in the confounding terms (our "free information" intervention). Alternatively, agents can be evaluated on the basis of a body of work (our "multiple signal" intervention). These observations are consistent with practices that have arisen in academic research, including the establishment of third-party funding agencies (e.g. the NSF) to support basic science and methodological research, and the evaluation of researchers based on advancements developed across several papers (e.g. tenure and various prizes).

We conclude below with brief mention of additional directions and interpretations of the model. So far we have focused on a sequence of decision-makers with a common prior. We might alternatively consider multiple "communities" of decision-makers, where decision-makers from the same community have the same prior, but priors differ across communities. This is in the spirit of Harel et al. (2018), which considers social learning on a network, and also Sethi and Yildiz (2017), which considers information acquisitions by individuals partitioned into different groups. Using this setup, our results can be interpreted as answering the question: *Will individuals from different communities end up observing the same (best) set of sources, or will they persistently acquire information from different sources?* Our main results show that when there is a unique strongly complementary set of sources, then different priors wash out; otherwise, different priors can result in persistent differences in what sources are listened to across groups.[29]

Second, our model considers the demand for information given an exogenous set of information sources. Another natural model would have the information sources choose the information they provide in order to maximize demand (see a related problem in Perego and Yuksel (2018).) Our characterization of the optimal frequency vector λ^* implies the following comparative static: If signal i is viewed with positive frequency in the optimal benchmark, then this frequency is (locally) decreasing in its precision. Thus, if demand is interpreted as λ^*_i (the long-run frequency with which source i is optimally viewed), sources face conflicting incentives: They want to provide information sufficiently precise to be included in the best set and receive viewership at all, but subject to this, they want to provide signals as im-

[29] In our main model with persistent states, opinions about ω end up converging across the population even if different groups frequent different sources. However, if we think that individuals choose which sources to listen to based on ω, but end up learning from these sources about other issues as well (e.g. if these are news sources), then different source-viewerships can result in large differences about beliefs regarding other matters.

precise as possible. These conflicting forces suggest that characterization of the equilibrium provisions of information precision may be subtle.

Finally, while we have described our setting as choice between information sources, our model may apply more generally to choice between actions with complementarities. For example, suppose a sequence of managers take actions that have externalities for future managers, and each manager seeks to maximize performance of the company during his tenure. The concepts we have developed here of *efficient information aggregation* and *learning traps* have natural analogues in that setting (actions that maximize the company's long-term welfare, versus those that do not). Relative to the general setting, we study here a class of complementarities that are micro-founded in correlated signals. It is an interesting question of whether and how the forces we find here generalize to other kinds of complementarities.

A Proofs for the Main Model

The structure of the appendix follows that of the paper. In this appendix we provide proofs for the results in our main model, where states are perfectly persistent. These results are proved in the same order as they appeared in the main text; the only exception is that the proof of part (b) of Proposition 2 relies on tools we develop in the other proofs, and so it is given at the end. The next appendix provides proofs for the autocorrelated model as discussed in Section 8. Other results and examples are deferred to a separate Online Appendix.

A.1 Preliminaries

A.1.1 Posterior Variance Function

Throughout, let C denote the $N \times K$ matrix of signal coefficients, whose i-th row is the vector c_i' associated with signal i. Here we review and extend a basic result from Liang, Mu and Syrgkanis (2017). Specifically, we show that the posterior variance about ω weakly decreases over time, and the marginal value of any signal decreases in its signal count.

Lemma 2. *Given prior covariance matrix Σ^0 and q_i observations of each signal i, society's posterior variance about ω is*

$$V(q_1, \dots, q_N) = \left[((\Sigma^0)^{-1} + C'QC)^{-1}\right]_{11} \tag{2}$$

where $Q = \operatorname{diag}(q_1, \dots, q_N)$. The function V is decreasing and convex in each q_i whenever these arguments take non-negative real values.

Proof. Note that $(\Sigma^0)^{-1}$ is the prior precision matrix and $C'QC = \sum_{i=1}^N q_i \cdot [c_i c_i']$ is the total precision from the observed signals. Thus (2) simply represents the fact that for Gaussian prior and signals, the posterior precision matrix is the sum of the prior and signal precision matrices. To prove the monotonicity of V, consider the partial order $\succeq$ on positive semi-definite matrices where $A \succeq B$ if and only if $A - B$ is positive semi-definite. As q_i increases, the matrix Q and $C'QC$ increase in this order. Thus the posterior covariance matrix $((\Sigma^0)^{-1} + C'QC)^{-1}$ decreases in this order, which implies that the posterior variance about ω decreases.

To prove that V is convex, it suffices to prove that V is *midpoint-convex* since the function is clearly continuous.[30] Take $q_1, \dots, q_N, r_1, \dots, r_N \in \mathbb{R}_+$ and let $s_i = \frac{q_i + r_i}{2}$. Define

[30] A function V is midpoint-convex if the inequality $V(a) + V(b) \geq 2V(\frac{a+b}{2})$ always holds. Every continuous function that is midpoint-convex is also convex.

the corresponding diagonal matrices to be Q, R, S. Note that $Q + R = 2S$. Thus by the AM-HM inequality for positive-definite matrices, we have

$$((\Sigma^0)^{-1} + C'QC)^{-1} + ((\Sigma^0)^{-1} + C'RC)^{-1} \succeq 2((\Sigma^0)^{-1} + C'SC)^{-1}.$$

Using (2), we conclude that

$$V(q_1, \dots, q_N) + V(r_1, \dots, r_N) \geq 2V(s_1, \dots, s_N).$$

This proves the (midpoint) convexity of V. □

A.1.2 Inverse of Positive Semi-definite Matrices

For future use, we provide a definition of $[X^{-1}]_{11}$ for positive *semi-definite* matrices X. When X is positive definite, its eigenvalues are strictly positive, and its inverse matrix is defined as usual. In general, we can apply the Spectral Theorem to write

$$X = UDU',$$

where U is a $K \times K$ orthogonal matrix whose columns are eigenvectors of X, and $D = \operatorname{diag}(d_1, \dots, d_K)$ is a diagonal matrix consisting of non-negative eigenvalues. Even if some of these eigenvalues are zero, we can think of X^{-1} as

$$X^{-1} = (UDU')^{-1} = UD^{-1}U' = \sum_{j=1}^{K} \frac{1}{d_j} \cdot [u_j u_j']$$

where u_j is the j-th column vector of U. We thus define

$$[X^{-1}]_{11} := \sum_{j=1}^{K} \frac{(\langle u_j, e_1 \rangle)^2}{d_j}, \tag{3}$$

with the convention that $\frac{0}{0} = 0$ and $\frac{z}{0} = \infty$ for any $z > 0$. Note that by this definition,

$$[X^{-1}]_{11} = \lim_{\epsilon \to 0_+} \left(\sum_{j=1}^{K} \frac{(\langle u_j, e_1 \rangle)^2}{d_j + \epsilon} \right) = [(X + \epsilon I_K)^{-1}]_{11},$$

since the matrix $X + \epsilon I_K$ has the same set of eigenvectors as X, with eigenvalues increased by ϵ. Hence our definition of $[X^{-1}]_{11}$ is a continuous extension of the usual definition to positive semi-definite matrices.

A.1.3 Asymptotic Posterior Variance

We can approximate the posterior variance as a function of the frequencies with which each signal is observed. Specifically, for any $\lambda \in \mathbb{R}^N_+$, define

$$V^*(\lambda) := \lim_{t\to\infty} t \cdot V(\lambda t).$$

The following result shows V^* to be well-defined and computes its value:

Lemma 3. *Let* $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_N)$. *Then*

$$V^*(\lambda) = [(C'\Lambda C)^{-1}]_{11} \tag{4}$$

The value of $[(C'\Lambda C)^{-1}]_{11}$ *is well-defined, see (3).*

Proof. Recall that $V(q_1, \ldots, q_N) = [((\Sigma^0)^{-1} + C'QC)^{-1}]_{11}$ with $Q = \mathrm{diag}(q_1, \ldots, q_N)$. Thus

$$t \cdot V(\lambda_1 t, \ldots, \lambda_N t) = \left[\left(\frac{1}{t}(\Sigma^0)^{-1} + C'\Lambda C\right)^{-1}\right]_{11}.$$

Hence the lemma follows from the continuity of $[X^{-1}]_{11}$ in the matrix X. □

We note that $C'\Lambda C$ is the Fisher Information Matrix when signals are observed according to frequencies λ. Thus the above lemma can also be seen as an application of the Bayesian Central Limit Theorem.

A.2 Key Object ϕ

We now define an object that will play a central role in the proofs. For each set of signals $\mathcal{S}$, consider writing the first coordinate vector $e_1 \in \mathbb{R}^K$ (corresponding to the payoff-relevant state ω) as a linear combination of signals in $\mathcal{S}$:

$$e_1 = \sum_{i\in\mathcal{S}} \beta_i^{\mathcal{S}} \cdot c_i.$$

Definition 4. $\phi(\mathcal{S}) := \min_\beta \sum_{i\in\mathcal{S}} |\beta_i^{\mathcal{S}}|$.

That is, $\phi(\mathcal{S})$ measures the size of the "smallest" (in the L-1 norm) linear combination of the signals in $\mathcal{S}$ to produce an unbiased estimate of the payoff-relevant state. In case ω is not spanned by $\mathcal{S}$, this definition sets $\phi(\mathcal{S}) = \infty$.

Note that when $\mathcal{S}$ *minimally spans* ω (so that no subset spans), the coefficients $\beta_i^{\mathcal{S}}$ are unique and nonzero. In this case $\phi(\mathcal{S})$ is easy to compute. In general, we have the following characterization:

Lemma 4. *For any set $\mathcal{S}$ that spans ω, $\phi(\mathcal{S}) = \min_{\mathcal{T} \subset \mathcal{S}} \phi(\mathcal{T})$ where the minimum is over subsets $\mathcal{T}$ that "minimally span" ω.*

This lemma is a standard result linear programming, so we omit the proof. We note that when $\mathcal{S}^*$ is the set of signals that minimally span ω and also minimize ϕ, we in particular have $\phi([N]) = \phi(\mathcal{S}^*)$. As the following proposition makes clear, this set $\mathcal{S}^*$ is exactly the best complementary set defined in the main text.

Proposition 6. *For any set of signals $\mathcal{S}$,* $\mathrm{val}(\mathcal{S}) = \frac{1}{\phi(\mathcal{S})^2}$.

Note that Liang, Mu and Syrgkanis (2017) proved this proposition for sets $\mathcal{S}$ that minimally span ω. We will prove this result in its general form shortly.

A.3 Proof of Proposition 1 and Claim 1

To see why Proposition 6 is useful, let us use it to show the results in Section 3 and 4. Indeed, Claim 1 directly follows from Proposition 6 and the definition of ϕ.

As for Proposition 1, take any set of signals $\mathcal{S}$. If these signals do not span ω, then Proposition 6 implies $\mathrm{val}(\mathcal{S}) = 0$ and $\mathcal{S}$ is not complementary by Definition 2. If a *proper* subset of $\mathcal{S}$ spans ω, then Proposition 6 together with Lemma 4 implies that the informational value of $\mathcal{S}$ is equal to the highest value among its subsets that minimally span ω. Let $\mathcal{S}'$ denote this subset that achieves this highest value. For this $\mathcal{S}'$ the inequality in Definition 2 is violated, and $\mathcal{S}$ is again not complementary.

Finally, suppose $\mathcal{S}$ itself minimally spans ω. In this case any nonempty proper subset of $\mathcal{S}$ does not span ω and have zero informational value, whereas $\mathcal{S}$ has positive value. So Definition 2 is satisfied and such sets $\mathcal{S}$ are complementary, as described in Proposition 1.

A.4 Proof of Proposition 6 and Proposition 2 Part (a)

We will focus on proving $\mathrm{val}([N]) = \frac{1}{\phi(\mathcal{S}^*)^2}$, which is thus equal to $\frac{1}{\phi([N])^2}$ by Lemma 4. Once this is proved, it is a direct generalization that $\mathrm{val}(\mathcal{S}) = \frac{1}{\phi(\mathcal{S})^2}$ whenever $\mathcal{S}$ spans ω. And in case $\mathcal{S}$ does not span ω, the posterior variance of ω is bounded away from zero when agents are constrained to observe from $\mathcal{S}$. Thus $\tau(q^t)$ is bounded and $\mathrm{val}(\mathcal{S}) = \limsup_{t\to\infty} \frac{\tau(q^t)-\tau_0}{t} = 0$, which is also equal to $\frac{1}{\phi(\mathcal{S})^2}$.

A.4.1 Reduction to Study of V^*

Consider the asymptotic posterior variance function V^* introduced previously. We claim that $\mathrm{val}([N]) = \frac{1}{\phi(\mathcal{S}^*)^2}$ will follow from the fact that λ^* is the (unique) frequency vector that minimizes V^*.

Lemma 5. *Suppose λ^* uniquely minimizes $V^*(\lambda)$ for $\lambda \in \Delta^{N-1}$. Then* $\mathrm{val}([N]) = \frac{1}{\phi(\mathcal{S}^*)^2}$*, and Proposition 6 as well as part (a) of Proposition 2 holds.*

Proof. By definition, we always have $\mathrm{val}([N]) \geq \mathrm{val}(\mathcal{S}^*) = \frac{1}{\phi(\mathcal{S}^*)^2}$. In the opposite direction, take any sequence q^t with $t \to \infty$ with $\limsup_t \frac{\tau(q^t)-\tau_0}{t} = \mathrm{val}([N])$. Since τ_0 is a constant, we equivalently have $\limsup_t \frac{\tau(q^t)}{t} = \mathrm{val}([N])$, which gives

$$\liminf_{t\to\infty} \; t \cdot V(q^t) = \frac{1}{\mathrm{val}([N])}$$

using the fact that the precision $\tau(q^t)$ is just the inverse of the variance $V(q^t)$.

By passing to a subsequence if necessary, we may assume the frequency vector $\lambda := \lim_{t\to\infty} \frac{q^t}{t}$ exists. Then by definition of V^*, the LHS of the above display is simply $V^*(\lambda)$. We therefore deduce $\mathrm{val}([N]) = \frac{1}{V^*(\lambda)}$ for some $\lambda \in \Delta^{N-1}$. Since λ^* minimizes V^*, we conclude that $\mathrm{val}([N]) \leq \frac{1}{V^*(\lambda^*)} = \frac{1}{\phi(\mathcal{S}^*)^2}$.

Combined with the earlier analysis, $\mathrm{val}([N]) = \frac{1}{\phi(\mathcal{S}^*)^2}$ must hold with equality. Moreover, since λ^* is the unique minimizer, equality can only hold when $\lambda = \lambda^*$, so that $\frac{q^t}{t}$ necessarily converges to λ^*. This is what we desire to prove for Proposition 2. □

A.4.2 Crucial Lemma

To show λ^* uniquely minimizes V^*, we need the following technical lemma.

Lemma 6. *Suppose $\mathcal{S}^*$ (which uniquely minimizes ϕ) involves exactly K signals, and without loss let $\mathcal{S}^* = \{1, \dots, K\}$. Let C^* be the $K \times K$ submatrix of C corresponding to the first K signals. Further suppose $\beta_i^{\mathcal{S}^*} = [(C^*)^{-1}]_{1i}$ is positive for $1 \leq i \leq K$. Then for any signal $j > K$, if we write $c_j = \sum_{i=1}^K \alpha_i \cdot c_i$ (which is a unique representation), then $|\sum_{i=1}^K \alpha_i| < 1$.*

Proof. By assumption, we have the vector identity

$$e_1 = \sum_{i=1}^{K} \beta_i \cdot c_i \quad \text{with } \beta_i = [(C^*)^{-1}]_{1i} > 0.$$

Suppose for contradiction that $\sum_{i=1}^K \alpha_i \geq 1$ (the opposite case where the sum is ≤ -1 can be similarly treated). Then some α_i must be positive. Without loss of generality, we assume $\frac{\alpha_1}{\beta_1}$ is the largest among such ratios. Then $\alpha_1 > 0$ and

$$e_1 = \sum_{i=1}^{K} \beta_i \cdot c_i = \left(\sum_{i=2}^{K} \left(\beta_i - \frac{\beta_1}{\alpha_1} \cdot \alpha_i \right) \cdot c_i \right) + \frac{\beta_1}{\alpha_1} \cdot \left(\sum_{i=1}^{K} \alpha_i \cdot c_i \right)$$

This represents e_1 as a linear combination of the vectors $c_2, \dots, c_K$ and c_j, with coefficients $\beta_2 - \frac{\beta_1}{\alpha_1} \cdot \alpha_2, \dots, \beta_K - \frac{\beta_1}{\alpha_1} \cdot \alpha_K$ and $\frac{\beta_1}{\alpha_1}$. Note that these coefficients are non-negative: For each

$2 \le i \le K$, $\beta_i - \frac{\beta_1}{\alpha_1} \cdot \alpha_i$ is clearly positive if $\alpha_i \le 0$ (since $\beta_i > 0$). And if $\alpha_i > 0$, $\beta_i - \frac{\beta_1}{\alpha_1} \cdot \alpha_i$ is again non-negative by the assumption that $\frac{\alpha_i}{\beta_i} \le \frac{\alpha_1}{\beta_1}$.

By definition, $\phi(\{2, \dots, K, j\})$ is the sum of the absolute value of these coefficients. This sum is

$$\sum_{i=2}^{K} \left(\beta_i - \frac{\beta_1}{\alpha_1} \cdot \alpha_i \right) + \frac{\beta_1}{\alpha_1} = \sum_{i=1}^{K} \beta_i + \frac{\beta_1}{\alpha_1} \cdot \left(1 - \sum_{i=1}^{K} \alpha_i \right) \le \sum_{i=1}^{K} \beta_i.$$

But then $\phi(\{2, \dots, K, j\}) \le \phi(\{1, 2, \dots, K\})$, contradicting the unique minimality of $\phi(\mathcal{S}^*)$. Hence the lemma must be true. □

A.4.3 Case 1: $|\mathcal{S}^*| = K$

In this section, we prove that λ^* is indeed the unique minimizer of V^* whenever the set $\mathcal{S}^*$ contains exactly K signals. Later on we will prove the same result even when $|\mathcal{S}^*| < K$, but that proof will require additional techniques. As discussed, this result will imply Proposition 6 and part (a) of Proposition 2.

Lemma 7. *The function $V^*(\lambda)$ is uniquely minimized at $\lambda = \lambda^*$.*

Proof. First, we assume $\mathcal{S}^* = \{1, \dots, K\}$ and that $[(C^*)^{-1}]_{1i}$ is positive for $1 \le i \le K$. This is without loss because we can replace c_i with $-c_i$ without affecting the model.

Since $V(q_1, \dots, q_N)$ is convex in its arguments, $V^*(\lambda) = \lim_{t \to \infty} t \cdot V(\lambda_1 t, \dots, \lambda_N t)$ is also convex in λ. To show λ^* uniquely minimizes V^*, we only need to show λ^* is a *local minimum*. In other words, it suffices to show $V^*(\lambda^*) < V^*(\lambda)$ for any λ that belongs to an ϵ-neighborhood of λ^*. By definition, $\mathcal{S}^*$ minimally spans ω and so its signals are linearly independent. Under the additional assumption that $\mathcal{S}^*$ has size K, we deduce that its signals span the entire space $\mathbb{R}^K$. From this it follows that the $K \times K$ matrix $C'\Lambda^* C$ is positive definite, and by (4) the function V^* is differentiable near λ^*.

We claim that the partial derivatives of V^* satisfy the following inequality:

$$\partial_K V^*(\lambda^*) < \partial_j V^*(\lambda^*) \le 0, \forall j > K. \tag{*}$$

Once this is proved, we will have, for λ close to λ^*,

$$V^*(\lambda_1, \dots, \lambda_K, \lambda_{K+1}, \dots, \lambda_N) \ge V^* \left(\lambda_1, \dots, \lambda_{K-1}, \sum_{k=K}^{N} \lambda_k, 0, \dots, 0 \right) \ge V^*(\lambda^*). \tag{5}$$

The first inequality is based on (*) and differentiability of V^*, while the second inequality is because λ^* uniquely minimizes V^* when restricting to the first K signals. Moreover, when $\lambda \neq \lambda^*$, one of these inequalities is strict so that $V^*(\lambda) > V^*(\lambda^*)$ holds strictly.

To prove (*), we recall that

$$V^*(\lambda) = e_1'(C'\Lambda C)^{-1}e_1.$$

Since $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_N)$, its derivative is $\partial_i \Lambda = \Delta_{ii}$, which is an $N \times N$ matrix whose (i,i)-th entry is 1 with all other entries equal to zero. Using properties of matrix derivatives, we obtain

$$\partial_i V^*(\lambda) = -e_1'(C'\Lambda C)^{-1}C'\Delta_{ii}C(C'\Lambda C)^{-1}e_1.$$

As the i-th row vector of C is c_i', $C'\Delta_{ii}C$ is the $K \times K$ matrix $c_i c_i'$. The above simplifies to

$$\partial_i V^*(\lambda) = -[e_1'(C'\Lambda C)^{-1}c_i]^2.$$

At $\lambda = \lambda^*$, the matrix $C'\Lambda C$ further simplifies to $(C^*)' \cdot \operatorname{diag}(\lambda_1^*, \ldots, \lambda_K^*) \cdot (C^*)$, which is a product of $K \times K$ invertible matrices. We thus deduce that

$$\partial_i V^*(\lambda^*) = -\left[e_1' \cdot (C^*)^{-1} \cdot \operatorname{diag}\left(\frac{1}{\lambda_1^*}, \ldots, \frac{1}{\lambda_K^*}\right) \cdot ((C^*)')^{-1} \cdot c_i\right]^2.$$

Crucially, note that the term in the brackets is a linear function of c_i. To ease notation, we write $v' = e_1' \cdot (C^*)^{-1} \cdot \operatorname{diag}\left(\frac{1}{\lambda_1^*}, \ldots, \frac{1}{\lambda_K^*}\right) \cdot ((C^*)')^{-1}$ and $\gamma_i = \langle v, c_i \rangle$. Then

$$\partial_i V^*(\lambda^*) = -\gamma_i^2, \ 1 \le i \le N. \tag{6}$$

For $1 \le i \le K$, $((C^*)')^{-1} \cdot c_i$ is just e_i. Thus, using the assumption $[(C^*)^{-1}]_{1i} > 0, \forall i$, we have

$$\gamma_i = e_1' \cdot (C^*)^{-1} \cdot \operatorname{diag}\left(\frac{1}{\lambda_1^*}, \ldots, \frac{1}{\lambda_K^*}\right) \cdot e_i = \frac{[(C^*)^{-1}]_{1i}}{\lambda_i^*} = \phi(\mathcal{S}^*). \tag{7}$$

On the other hand, choosing any signal $j > K$, we can uniquely write the vector c_j as a linear combination of $c_1, \ldots, c_K$. By Lemma 6,

$$\gamma_j = \langle v, c_j \rangle = \sum_{i=1}^{K} \alpha_i \cdot \langle v, c_i \rangle = \sum_{i=1}^{K} \alpha_i \cdot \gamma_i = \phi(\mathcal{S}^*) \cdot \sum_{i=1}^{K} \alpha_i, \tag{8}$$

where the last equality uses (7). Since $|\sum_{i=1}^K \alpha_i| < 1$, the absolute value of γ_j is strictly smaller than the absolute value of γ_K for any $j > K$. This together with (6) proves the desired inequality (*), and Lemma 7 follows. □

A.4.4 A Perturbation Argument

To summarize, we have shown that when ϕ is uniquely minimized by a set $\mathcal{S}$ containing exactly K signals,

$$\min_{\lambda \in \Delta^{N-1}} V^*(\lambda) = V^*(\lambda^*) = \phi(\mathcal{S}^*)^2 = \phi([N])^2.$$

We now use a perturbation argument to show this equality holds more generally.

Lemma 8. *For any coefficient matrix C,*

$$\min_{\lambda \in \Delta^{N-1}} V^*(\lambda) = \phi([N])^2. \tag{9}$$

Proof. For general coefficient matrix C, the set $\mathcal{S}$ that minimizes ϕ may not be unique or involve K signals. However, since society can choose to focus on $\mathcal{S}$, we always have

$$\min_{\lambda} V^*(\lambda) \leq V^*(\lambda^*) = \phi(\mathcal{S}^*)^2 = \phi([N])^2.$$

It remains to prove $V^*(\lambda) \geq \phi([N])^2$ for every $\lambda \in \Delta^{N-1}$. By Lemma 3, we need to show $[(C'\Lambda C)^{-1}]_{11} \geq \phi([N])^2$.

Note that we already proved this inequality for *generic* coefficient matrices C: specifically, those for which $\phi(\mathcal{S})$ is uniquely minimized by a set of K signals. But even if C is "non-generic", we can approximate it by a sequence of generic matrices C_m.[31] Along this sequence, we have

$$[(C_m'\Lambda C_m)^{-1}]_{11} \geq \phi_m([N])^2$$

where ϕ_m is the analogue of ϕ for the coefficient matrix C_m.

As $m \to \infty$, the LHS above approaches $[(C'\Lambda C)^{-1}]_{11}$. We will show that on the RHS $\limsup_{m\to\infty} \phi_m([N]) \geq \phi([N])$, which would imply $[(C'\Lambda C)^{-1}]_{11} \geq \phi([N])^2$ and the lemma. Indeed, suppose $e_1 = \sum_i \beta_i^{(m)} \cdot c_i^{(m)}$ along the convergent sequence, then $e_1 = \sum_i \beta_i \cdot c_i$ for any limit point β of $\beta^{(m)}$. Using the definition of ϕ, this enables us to conclude $\liminf_{m\to\infty} \phi_m([N]) \geq \phi([N])$, which is more than sufficient. □

A.4.5 Case 2: $|\mathcal{S}^*| < K$

We now consider the case where $\mathcal{S}^* = \{1, \dots, k\}$ with $k < K$. We will show that λ^* is still the unique minimizer of $V^*(\cdot)$. Since $V^*(\lambda^*) = \phi(\mathcal{S}^*)^2 = \phi([N])^2$ by definition, we know from Lemma 8 that λ^* does minimize V^*. It remains to show λ^* is the *unique* minimizer.

To do this, we will consider a perturbed informational environment in which signals $k+1, \dots, N$ are made slightly more precise. Specifically, let $\eta > 0$ be a small positive number. Consider an alternative signal coefficient matrix $\tilde{C}$ with $\tilde{c_i} = c_i$ for $i \leq k$ and $\tilde{c_i} = (1+\eta)c_i$ for $i > k$. Let $\tilde{\phi}(\mathcal{S})$ be the analogue of ϕ for this alternative environment. It is clear that $\tilde{\phi}(\mathcal{S}^*) = \phi(\mathcal{S}^*)$, while $\tilde{\phi}(\mathcal{S})$ is slightly smaller than $\phi(\mathcal{S})$ for $\mathcal{S} \neq \mathcal{S}^*$. Thus with sufficiently small η, the set $\mathcal{S}^*$ remains the unique minimizer of ϕ (among sets that minimally span ω) in this perturbed environment, and the definition of λ^* is also maintained.

[31] First, we may add repetitive signals to ensure $N \geq K$. This does not affect the value of $\min_\lambda V^*(\lambda)$ or $\phi([N])$. Whenever $N \geq K$, it is generically true that every set that minimally spans ω contains exactly K signals. Moreover, the equality $\phi(\mathcal{S}) = \phi(\tilde{\mathcal{S}})$ for $\mathcal{S} \neq \tilde{\mathcal{S}}$ induces a non-trivial polynomial equation over the entries in C. This means we can always find C_m close to C such that for each coefficient matrix C_m, different subsets $\mathcal{S}$ of size K attain different values of ϕ, so that ϕ is uniquely minimized.

Let $\tilde{V}^*$ be the perturbed asymptotic posterior variance function, then our previous analysis shows that $\tilde{V}^*$ has minimum value $\phi(\mathcal{S}^*)^2$ on the simplex. Taking advantage of the connection between V^* and $\tilde{V}^*$, we thus have

$$\begin{aligned} V^*(\lambda_1, \ldots, \lambda_N) &= \tilde{V}^* \left(\lambda_1, \ldots, \lambda_k, \frac{\lambda_{k+1}}{(1+\eta)^2}, \ldots, \frac{\lambda_N}{(1+\eta)^2} \right) \\ &\geq \frac{\phi(\mathcal{S}^*)^2}{\sum_{i \leq k} \lambda_i + \frac{1}{(1+\eta)^2} \sum_{i>k} \lambda_i}. \end{aligned}$$

The equality uses (4) and $C'\Lambda C = \sum_i \lambda_i c_i c_i' = \sum_{i \leq k} \lambda_i c_i c_i' + \sum_{i>k} \frac{\lambda_i}{(1+\eta)^2} \tilde{c}_i \tilde{c}_i{}'$. The inequality follows from the homogeneity of $\tilde{V}^*$.

The above display implies that any frequency vector λ,

$$V^*(\lambda) \geq \frac{\phi(\mathcal{S}^*)^2}{1 - \frac{2\eta+\eta^2}{(1+\eta)^2} \sum_{i>k} \lambda_i} \geq \frac{\phi(\mathcal{S}^*)^2}{1 - \eta \sum_{i>k} \lambda_i} \quad \text{for some } \eta > 0. \tag{10}$$

Hence $V^*(\lambda) > \phi(\mathcal{S}^*)^2 = V^*(\lambda^*)$ whenever λ puts positive weight outside of $\mathcal{S}^*$. But it is easily checked that $V^*(\lambda)$ is uniquely minimized at λ^* when λ is supported on $\mathcal{S}^*$. Hence λ^* is the unique minimizer of V^* over the whole simplex. This proves Lemma 7, which completes the proof of the propositions via Lemma 5.

A.5 Proof of Theorem 1 Part (a)

Let signals $1, \ldots, k$ (with $k \leq K$) be a strongly complementary set; by Lemma 1 in the main text, these signals are best in their subspace. We will demonstrate an open set of prior beliefs given which *all agents* observe these k signals. Since these signals are complementary, Proposition 1 implies they must be linearly independent. Thus we can consider linearly transformed states $\tilde{\theta}_1, \ldots, \tilde{\theta}_K$ such that these k signals are simply $\tilde{\theta}_1, \ldots, \tilde{\theta}_k$ plus standard Gaussian noise. This linear transformation is invertible, so any prior over the original states is bijectively mapped to a prior over the transformed states. Thus it is without loss to work with the transformed model and look for prior beliefs over the transformed states.

The payoff-relevant state ω becomes a linear combination $\lambda_1^* \tilde{\theta}_1 + \cdots + \lambda_k^* \tilde{\theta}_k$ (up to a scalar multiple). Since the first k signals are best in their subspace, Lemma 6 before implies that any other signal belonging to this subspace can be written as

$$\sum_{i=1}^{k} \alpha_i \tilde{\theta}_i \; + \; \mathcal{N}(0,1)$$

with $|\sum_{i=1}^k \alpha_i| < 1$. On the other hand, if a signal does not belong to this subspace, it must take the form of

$$\sum_{i=1}^{K} \beta_i \tilde{\theta}_i \; + \; \mathcal{N}(0,1)$$

with $\beta_{k+1}, \dots, \beta_K$ not all equal to zero.

Now consider any prior belief with precision matrix P; the inverse of P is the prior covariance matrix (in terms of the transformed states). Suppose ϵ is a very small positive number, and P satisfies the following conditions:

1. For $1 \le i \le k$, $P_{ii} \ge \frac{1}{\epsilon^2}$;
2. For $1 \le i \ne j \le k$, $\frac{P_{ii}}{\lambda_i^*} \le (1+\epsilon) \cdot \frac{P_{jj}}{\lambda_j^*}$;
3. For $k+1 \le i \le K$, $P_{ii} \in [\epsilon, 2\epsilon]$;
4. For $1 \le i \ne j \le K$, $|P_{ij}| \le \epsilon^2$.

It is clear that any such P is positive definite, since on each row the diagonal entry has dominant size. Moreover, the set of P is open. Below we show that given any such prior, the myopic signal choice is among the first k signals, and that the posterior precision matrix also satisfies the same four conditions. As such, *all* agents would choose from the first k signals.

Let $V = P^{-1}$ be the prior covariance matrix. Applying Cramer's rule for the matrix inverse, the above conditions on P imply the following conditions on V:

1. For $1 \le i \le k$, $V_{ii} \le 2\epsilon^2$;
2. For $1 \le i \ne j \le k$, $V_{ii}\lambda_i^* \le (1+L\epsilon) \cdot V_{jj}\lambda_j^*$;
3. For $k+1 \le i \le K$, $V_{ii} \in [\frac{1}{4\epsilon}, \frac{2}{\epsilon}]$;
4. For $1 \le i \ne j \le K$, $|V_{ij}| \le L\epsilon \cdot V_{ii}$.

Here L is a constant depending only on K (but not on ϵ). For example, the last condition is equivalent to $\det(P_{-ij}) \le L\epsilon \cdot \det(P_{-ii})$. This is proved by expanding both determinants into multilinear sums, and using the fact that on each row of P the off-diagonal entries are at most ϵ-fraction of the diagonal entry.

Given this matrix V, the variance reduction of $\omega = \sum_{i=1}^k \lambda_i^* \tilde{\theta}_i$ by any signal $\sum_{i=1}^k \alpha_i \tilde{\theta}_i + \mathcal{N}(0,1)$ can be computed as

$$\frac{(\sum_{i,j=1}^k \alpha_i \lambda_j^* V_{ij})^2}{1 + \sum_{i,j=1}^k \alpha_i \alpha_j V_{ij}},$$

where the denominator is the variance of the signal and the numerator is the covariance between the signal and ω. By the first and last conditions on V, the denominator here is $1 + O(\epsilon^2)$. By the second and last condition, the numerator is

$$\left((\sum_{i=1}^k \alpha_i + O(\epsilon)) \cdot \lambda_1^* V_{11} \right)^2.$$

Since $|\sum_{i=1}^{k} \alpha_i| < 1$, we deduce that any other signal belonging to the subspace of the first k signals is myopically worse than signal 1, whose variance reduction is $\frac{(\lambda_1^* V_{11})^2}{1+V_{11}}$.

Meanwhile, take any signal outside of the subspace. The variance reduction by such a signal $\sum_{i=1}^{K} \beta_i \tilde{\theta}_i + \mathcal{N}(0,1)$ is

$$\frac{(\sum_{i=1}^{K} \sum_{j=1}^{k} \beta_i \lambda_j^* V_{ij})^2}{1 + \sum_{i,j=1}^{K} \beta_i \beta_j V_{ij}}$$

By the second and last condition on V, the numerator here is $O((\lambda_1^* V_{11})^2)$. If we can show that the denominator is very large, then such a signal would also be myopically worse than signal 1. Indeed, since $V_{ij} = O(\epsilon^2)$ whenever $i \le k$ or $j \le k$, it is sufficient to show $\sum_{i,j>k} \beta_i \beta_j V_{ij}$ is large. This holds by the last two conditions on V and the assumption that $\beta_{k+1}, \dots, \beta_K$ are not all zero.[32]

Hence, we have shown that given any prior precision matrix P satisfying the above conditions, the myopic signal choice is among the first k signals. It remains to check the resulting posterior precision matrix $\hat{P}$ also satisfies those four conditions. If the signal acquired is signal i ($1 \le i \le k$), then $\hat{P} = P + \Delta_{ii}$. Therefore we only need to show the second condition holds for $\hat{P}$; that is, $\frac{P_{ii}+1}{\lambda_i^*} \le (1+\epsilon) \cdot \frac{P_{jj}}{\lambda_j^*}$ for each $1 \le j \le k$. To this end, we note that since signal i is myopically best given V, the following must hold:

$$\frac{(\lambda_i^* V_{ii})^2}{1 + V_{ii}} \ge \frac{(\lambda_j^* V_{jj})^2}{1 + V_{jj}}.$$

As $0 \le V_{ii}, V_{jj} \le 2\epsilon^2$, this implies $\lambda_i^* V_{ii} \ge (1-\epsilon^2)\lambda_j^* V_{jj}$. Now applying Cramer's rule to $V = P^{-1}$ again, we can deduce $V_{ii} = \frac{1+O(\epsilon^2)}{P_{ii}}$. So for ϵ small it holds that $\frac{P_{ii}}{\lambda_i^*} \le (1+\frac{\epsilon}{2}) \cdot \frac{P_{jj}}{\lambda_j^*}$. As $P_{ii} \ge \frac{1}{\epsilon^2}$, we also have $\frac{1}{\lambda_i^*} \le \frac{\epsilon}{2} \cdot \frac{P_{jj}}{\lambda_j^*}$. Adding up these two inequalities yields the second condition for $\hat{P}$ and completes the proof.

A.6 Proof of Theorem 1 Part (b)

A.6.1 Restated Version

Given any prior belief, let $\mathcal{A} \subset [N]$ be the set of all signals that are observed by infinitely many agents. We first show that $\mathcal{A}$ spans ω.

Indeed, by definition we can find some period t after which agents *exclusively* observe signals from $\mathcal{A}$. Note that the variance reduction of any signal approaches zero as its signal

[32] Formally, we can without loss assume $\beta_K^2 V_{KK}$ is largest among $\beta_i^2 V_{ii}$ for $i > k$. Then for any $i \ne j$, the last condition implies

$$\beta_i \beta_j V_{ij} \ge -L\epsilon \cdot \beta_i \beta_j \sqrt{V_{ii} V_{jj}} \ge -L\epsilon \cdot \beta_K^2 V_{KK}.$$

This trivially also holds for $i = j \ne K$. Summing across all pairs $(i,j) \ne (K,K)$ yields $\sum_{i,j>k} \beta_i \beta_j V_{ij} > (1 - K^2 L\epsilon)\beta_K^2 V_{KK}$, which must be large by the third condition on V.

count gets large. Thus, along society's signal path, the variance reduction is close to zero at sufficiently late periods. If $\mathcal{A}$ does not span ω, society's posterior variance remains bounded away from zero. Thus in the limit where each signal in $\mathcal{A}$ has infinite signal counts, there still exists some signal j outside of $\mathcal{A}$ whose variance reduction is strictly positive.[33] By continuity, we deduce that at any sufficiently late period, observing signal j is better than observing any signal in $\mathcal{A}$. This contradicts our assumption that later agents only observe signals in $\mathcal{A}$.

Now that $\mathcal{A}$ spans ω, we can take $\mathcal{S}$ to be the best complementary set in the subspace spanned by $\mathcal{A}$; $\mathcal{S}$ is strongly complementary by Lemma 1. To prove Theorem 1 part (b), we will show that long-run frequencies are positive precisely for the signals in $\mathcal{S}$. By ignoring the initial periods, we can assume without loss that *only signals in* $\overline{\mathcal{A}}$ *are available*. It thus suffices to show that whenever the signals observed infinitely often span a subspace, agents eventually focus on the best complementary set $\mathcal{S}$ in that subspace. To ease notation, we assume this subspace is the entire $\mathbb{R}^K$, and prove the following result:

Theorem 1 part (b) Restated. *Suppose that the signals observed infinitely often span* $\mathbb{R}^K$. *Then society's long-run frequency is* λ^*.

The next sections are devoted to the proof of this restatement.

A.6.2 Estimates of Derivatives

We introduce a few technical lemmata:

Lemma 9. *For any* $q_1, \dots, q_N$, *we have*

$$\left| \frac{\partial_{jj} V(q_1, \dots, q_N)}{\partial_j V(q_1, \dots, q_N)} \right| \le \frac{2}{q_j}.$$

Proof. Recall that $V(q_1, \dots, q_N) = e_1' \cdot [(\Sigma^0)^{-1} + C'QC]^{-1} \cdot e_1$. Thus

$$\partial_j V = -e_1' \cdot [(\Sigma^0)^{-1} + C'QC]^{-1} \cdot c_j \cdot c_j' \cdot [(\Sigma^0)^{-1} + C'QC]^{-1} \cdot e_1,$$

and

$$\partial_{jj} V = 2e_1' \cdot [(\Sigma^0)^{-1} + C'QC]^{-1} \cdot c_j \cdot c_j' \cdot [(\Sigma^0)^{-1} + C'QC]^{-1} \cdot c_j \cdot c_j' \cdot [(\Sigma^0)^{-1} + C'QC]^{-1} \cdot e_1.$$

[33] To see this, let $s_1, \dots, s_N$ denote the limit signal counts, where $s_i = \infty$ if and only if $i \in \mathcal{A}$. We need to find some signal j such that $V(s_j + 1, s_{-j}) < V(s_j, s_{-j})$. If such a signal does not exist, then all partial derivatives of V at s are zero. Since V is always differentiable (unlike V^*), this would imply that all directional derivatives of V are also zero. By the convexity of V, V must be minimized at s. However, the minimum value of V is zero because there exists a complementary set. This contradicts $V(s) > 0$.

Let $\gamma_j = e_1' \cdot [(\Sigma^0)^{-1} + C'QC]^{-1} \cdot c_j$, which is a number. Then the above becomes

$$\partial_j f = -\gamma_j^2; \qquad \partial_{jj} f = 2\gamma_j^2 \cdot c_j' \cdot [(\Sigma^0)^{-1} + C'QC]^{-1} \cdot c_j.$$

Note that $(\Sigma^0)^{-1} + C'QC \succeq q_j \cdot c_j c_j'$ in matrix norm. Thus the number $c_j' \cdot [(\Sigma^0)^{-1} + C'QC]^{-1} \cdot c_j$ is bounded above by $\frac{1}{q_j}$.[34] This proves the lemma. □

Since the second derivative is small compared to the first derivative, we deduce that the variance reduction of any *discrete* signal can be approximated by the partial derivative of f. This property is summarized in the following lemma:

Lemma 10. *For any $q_1, \dots, q_N$, we have*[35]

$$V(q) - V(q_j + 1, q_{-j}) \geq \frac{q_j}{q_j + 1} |\partial_j V(q)|.$$

Proof. We will show the more general result:

$$V(q) - V(q_j + x, q_{-j}) \geq \frac{q_j x}{q_j + x} \cdot |\partial_j V(q)|, \forall x \geq 0.$$

This clearly holds at $x = 0$. Differentiating with respect to x, we only need to show

$$-\partial_j V(q_j + x, q_{-j}) \geq \frac{q_j^2}{(q_j + x)^2} |\partial_j V(q)|, \forall x \geq 0.$$

Equivalently, we need to show

$$-(q_j + x)^2 \cdot \partial_j V(q_j + x, q_{-j}) \geq -q_j^2 \cdot \partial_j V(q), \forall x \geq 0.$$

Again, this inequality holds at $x = 0$. Differentiating with respect to x, it becomes

$$-2(q_j + x) \cdot \partial_j V(q_j + x, q_{-j}) - (q_j + x)^2 \cdot \partial_{jj} V(q_j + x, q_{-j}) \geq 0.$$

This is exactly the result of Lemma 9. □

[34]Formally, we need to show that for any $\epsilon > 0$, the number $c_j'[c_j c_j' + \epsilon I_K]^{-1} c_j$ is at most 1. Using the identify $Trace(AB) = Trace(BA)$, we can rewrite this number as

$$Trace([c_j c_j' + \epsilon I_K]^{-1} c_j c_j') = Trace(I_K - [c_j c_j' + \epsilon I_K]^{-1} \epsilon I_K) = K - \epsilon \cdot Trace([c_j c_j' + \epsilon I_K]^{-1}).$$

The matrix $c_j c_j'$ has rank 1, so $K - 1$ of its eigenvalues are zero. Thus the matrix $[c_j c_j' + \epsilon I_K]^{-1}$ has eigenvalue $1/\epsilon$ with multiplicity $K - 1$, and the remaining eigenvalue is positive. This implies $\epsilon \cdot Trace([c_j c_j' + \epsilon I_K]^{-1}) > K - 1$, and then the above display yields $c_j' \cdot [(\Sigma^0)^{-1} + C'QC]^{-1} \cdot c_j < 1$ as desired.

[35]Note that the convexity of V gives $V(q) - V(q_j + 1, q_{-j}) \leq |\partial_j V(q)|$. This lemma provides a converse that we need for the subsequent analysis.

A.6.3 Lower Bound on Variance Reduction

Our next result gives a lower bound on the directional derivative of V along the "optimal" direction λ^*:

Lemma 11. *For any $q_1, \dots, q_N$, we have*

$$|\partial_{\lambda^*} V(q)| \geq \frac{V(q)^2}{\phi(\mathcal{S}^*)^2}.$$

Proof. To compute this directional derivative, we think of agents acquiring signals in fractional amounts, where a fraction of a signal is just the same signal with precision multiplied by that fraction. Consider an agent who draws λ_i^* realizations of each signal i. Then he essentially obtains the following signals:

$$Y_i = \langle c_i, \theta \rangle + \mathcal{N}\left(0, \frac{1}{\lambda_i^*}\right), \forall i.$$

This is equivalent to

$$\lambda_i^* Y_i = \langle \lambda_i^* c_i, \theta \rangle + \mathcal{N}(0, \lambda_i^*), \forall i.$$

Such an agent receives at least as much information as the sum of these signals:

$$\sum_i \lambda_i^* Y_i = \sum_i \langle \lambda_i^* c_i, \theta \rangle + \sum_i \mathcal{N}(0, \lambda_i^*) = \frac{\omega}{\phi(\mathcal{S}^*)} + \mathcal{N}(0, 1).$$

Hence the agent's posterior precision about ω (which is the inverse of his posterior variance V) must increase by at least $\frac{1}{\phi(\mathcal{S}^*)^2}$ along the direction λ^*. The chain rule of differentiation yields the lemma. □

We can now bound the variance reduction at late periods:

Lemma 12. *Fix any $q_1, \dots, q_N$. Suppose L is a positive number such that $(\Sigma^0)^{-1} + C'QC \succeq Lc_j c_j'$ holds for each signal $j \in \mathcal{S}^*$. Then we have*

$$\min_{j \in \mathcal{S}^*} V(q_j + 1, q_{-j}) \leq V(q) - \frac{L}{L+1} \cdot \frac{V(q)^2}{\phi(\mathcal{S}^*)^2}.$$

Proof. Fix any signal $j \in \mathcal{S}^*$. Using the condition $(\Sigma^0)^{-1} + C'QC \succeq Lc_j c_j'$, we can deduce the following variant of Lemma 10:[36]

$$V(q) - V(q_j + 1, q_{-j}) \geq \frac{L}{L+1} |\partial_j V(q)|.$$

[36]Even though we are not guaranteed $q_j \geq L$, we can modify the prior and signal counts such that the precision matrix $(\Sigma^0)^{-1} + C'QC$ is unchanged, and signal j has been observed at least L times. This is possible thanks to the condition $(\Sigma^0)^{-1} + C'QC \succeq Lc_j c_j'$. Then, applying Lemma 10 to this modified problem yields the result here.

Since V is always differentiable, $\partial_{\lambda^*} V(q)$ is a convex combination of the partial derivatives of V.[37] Thus

$$\max_{j \in \mathcal{S}^*} |\partial_j V(q)| \geq |\partial_{\lambda^*} V(q)|$$

These inequalities together with Lemma 11 complete the proof. □

A.6.4 Proof of the Restated Theorem 1 Part (b)

We will show $t \cdot V(m(t)) \to \phi(\mathcal{S}^*)^2$, so that society eventually approximates the optimal speed of learning. Since λ^* is the unique minimizer of V^*, this will imply the desired conclusion $\frac{m(t)}{t} \to \lambda^*$ (e.g., via the second half of Proposition 2 part (a)).

To estimate $V(m(t))$, we note that for any fixed L, society's acquisitions $m(t)$ eventually satisfy the condition $(\Sigma^0)^{-1} + C'QC \succeq Lc_j c_j'$. This is due to our assumption that the signals observed infinitely often span $\mathbb{R}^K$, which implies that $C'QC$ becomes arbitrarily large in matrix norm. Hence, we can apply Lemma 12 to find that

$$V(m(t+1)) \leq V(m(t)) - \frac{L}{L+1} \cdot \frac{V(m(t))^2}{\phi(\mathcal{S}^*)^2}$$

for all $t \geq t_0$, where t_0 depends only on L.

We introduce the auxiliary function $g(t) = \frac{V(m(t))}{\phi(\mathcal{S}^*)^2}$. Then the above simplifies to

$$g(t+1) \leq g(t) - \frac{L}{L+1} g(t)^2.$$

Inverting both sides, we have

$$\frac{1}{g(t+1)} \geq \frac{1}{g(t)(1 - \frac{L}{L+1} g(t))} = \frac{1}{g(t)} + \frac{\frac{L}{L+1}}{1 - \frac{L}{L+1} g(t)} \geq \frac{1}{g(t)} + \frac{L}{L+1}. \tag{11}$$

This holds for all $t \geq t_0$. Thus by induction, $\frac{1}{g(t)} \geq \frac{L}{L+1}(t - t_0)$ and so $g(t) \leq \frac{L+1}{L(t-t_0)}$. Going back to the posterior variance function V, this implies

$$V(m(t)) \leq \frac{L+1}{L} \cdot \frac{\phi(\mathcal{S}^*)^2}{t - t_0}. \tag{12}$$

Hence, by choosing L sufficiently large in the first place and then considering large t, we find that society's speed of learning is arbitrarily close to the optimal speed $\phi(\mathcal{S}^*)^2$. This completes the proof.

We comment that the above argument leaves open the possibility that some signals outside of $\mathcal{S}^*$ are observed *infinitely often*, yet with *zero long-run frequency*. In Appendix B.1, we show this does not happen.

[37] While this may be a surprising contrast with V^*, the difference arises because the formula for V always involves a full-rank prior covariance matrix, whereas its asymptotic variant V^* corresponds to a flat prior.

A.7 Proof of Proposition 3

Given any history of observations, an agent can always allocate his B observations as follows: He draws $\lfloor B \cdot \lambda_i^* \rfloor$ realizations of each signal i, and samples arbitrarily if there is any capacity remaining. Here $\lfloor \ \rfloor$ denotes the floor function.

Fix any $\epsilon > 0$. If B is sufficiently large, then the above strategy acquires at least $(1-\epsilon) \cdot B \cdot \lambda_i^*$ observations of each signal i. Adapting the proof of Lemma 11, we see that the agent's posterior precision about ω must increase by $\frac{(1-\epsilon)B}{\phi(\mathcal{S}^*)^2}$ under this strategy. Thus the same must hold for his optimal strategy, so that society's posterior precision at time t is at least $\frac{(1-\epsilon)Bt}{\phi(\mathcal{S}^*)^2}$. This implies that average precision per signal is at least $\frac{1-\epsilon}{\phi(\mathcal{S}^*)^2}$, which can be arbitrarily close to the optimal precision $\text{val}([N]) = \frac{1}{\phi(\mathcal{S}^*)^2}$ with appropriate choice of ϵ.

Since λ^* is the unique minimizer of V^*, society's long-run frequencies must be close to λ^*. In particular, with ϵ sufficiently small, we can ensure that each signal in $\mathcal{S}^*$ are observed with positive frequencies. The restated Theorem 1 part (b) extends to the current setting and implies that society's long-run frequency must be λ^*. This yields the proposition.[38]

A.8 Proof of Proposition 4

Suppose without loss that the best complementary set $\mathcal{S}^*$ is $\{1, \ldots, k\}$. By taking a linear transformation, we further assume each of the first k signals only involves ω and the first $k-1$ confounding terms $b_1, \ldots, b_{k-1}$. We will show that whenever $k-1$ sufficiently precise signals are provided about each of these confounding terms, long-run frequency will converge to λ^* regardless of the prior belief.

Fix any positive real number L. Since the $k-1$ free signals are very precise, it is as if the prior precision matrix (after taking into account these free signals) satisfies

$$(\Sigma^0)^{-1} \succeq L^2 \sum_{i=2}^{k} \Delta_{ii}$$

where Δ_{ii} is the $K \times K$ matrix that has one at the (i,i) entry and zero otherwise. Recall also that society eventually learns ω. Thus at some late period t_0, society's acquisitions must satisfy

$$C'QC \succeq L^2 \Delta_{11}.$$

[38]This proof also suggests that how small ϵ (and how large B) need to be depends on the distance between the optimal speed of learning and the "second-best" speed of learning from any other complementary set. Intuitively, in order to achieve long-run efficient learning, agents need to allocate B observations in the best set to approximate the optimal frequencies. If another set of signals offers a speed of learning that is only slightly worse, we will need B sufficiently large for the approximately optimal frequencies in the best set to beat this other set.

Adding up the above two displays, we have

$$(\Sigma^0)^{-1} + C'QC \succeq L^2 \sum_{i=1}^{k} \Delta_{ii} \succeq L c_j c_j', \forall 1 \leq j \leq k.$$

The last inequality uses the fact that each c_j only involves the first k coordinates.

Now this is exactly the condition we need in order to apply Lemma 12: Crucially, whether or not the condition is met for signals j outside of $\mathcal{S}^*$ does not affect the argument there. Thus we can follow the proof of the restated Theorem 1 part (b) to deduce (12). That is, for fixed L and corresponding free information, society's long-run precision per signal is at least $\frac{L}{(L+1)\phi(\mathcal{S}^*)^2}$. This can be made arbitrarily close to the optimal average precision.

Identical to the previous proof, we deduce that for large L, society's long-run frequency must be close to λ^*. The restated Theorem 1 part (b) allows us to conclude that the frequency is exactly λ^*.

A.9 Proof of Proposition 2 Part (b)

We will first generalize part (b) of Theorem 1 to show that for *any* $\delta \in (0,1)$ and any prior belief, the Social Planner's sampling strategy that maximizes δ-discounted payoff yields frequency vectors that converge over time. Moreover, the limit is the optimal frequency vector associated with some strongly complementary set. Later we will argue that that for δ close to 1, this long-run outcome must be the best complementary set $\mathcal{S}^*$ from all priors.

A.9.1 Long-run Characterization for All δ

Here we prove the following result:

Proposition 7. *Suppose $\delta \in (0,1)$. Given any prior, let $d_\delta(t)$ denote the vector of signals counts associated with any signal acquisition strategy that maximizes the δ-discounted average payoff. Then $\lim_{t\to\infty} \frac{d_\delta(t)}{t}$ exists and is equal to $\lambda^{\mathcal{S}}$ for some strongly complementary set $\mathcal{S}$.*

Proof. We follow the proof of Theorem 1 part (b) in Appendix A.6. The same argument there shows that for any $\delta < 1$, any strategy that maximizesδ-discounted payoff must infinitely observe a set of signals that span ω. Therefore it remains to prove the analogue of the restated version of Theorem 1 part (b).

To do that, let

$$W(t) = (1-\delta) \sum_{t' \geq t} \delta^{t'-t} \cdot V(d(t'))$$

denote the expected discounted loss from period t onwards; henceforth we fix δ and use $d(t)$ as shorthand for $d_\delta(t)$. Suppose signal acquisitions in the first t periods satisfy $C'QC \succeq L c_j c_j'$

for each signal $j \in \mathcal{S}^*$, where L is some positive constant. Then we are going to show that

$$\frac{1}{W(t+1)} \geq \frac{1}{W(t)} + \frac{L}{(L+1)\phi(\mathcal{S}^*)^2}. \tag{13}$$

Once this is proved, we can choose L large to show $W(t) \leq \frac{(1+\epsilon)\phi(\mathcal{S}^*)^2}{t}$ for any $\epsilon > 0$ and all sufficiently large t. Pick m so that $\delta^m \leq \epsilon$. Then for $t' \in (t, t+m)$ we have $V(d(t')) \geq \frac{(1-\epsilon/2)\phi(\mathcal{S}^*)^2}{t'} \geq \frac{(1-\epsilon)\phi(\mathcal{S}^*)^2}{t}$, so that

$$(1-\delta)\sum_{t'=t+1}^{t+m-1} \delta^{t'-t} \cdot V(d(t')) \geq (\delta - \delta^m) \cdot \frac{(1-\epsilon)\phi(\mathcal{S}^*)^2}{t} \geq \frac{(\delta-\epsilon)(1-\epsilon)\phi(\mathcal{S}^*)^2}{t}.$$

Subtracting this from $W(t)$, we obtain

$$(1-\delta) \cdot V(d(t)) \leq \frac{(1+\epsilon-(\delta-\epsilon)(1-\epsilon))\,\phi(\mathcal{S}^*)^2}{t}$$

again for t sufficiently large depending on ϵ. Since ϵ is arbitrary, we would be able to conclude $t \cdot V(d(t)) \to \phi(\mathcal{S}^*)^2$, and $\frac{d(t)}{t} \to \lambda^*$ would follow.

To prove (13), we consider a deviation strategy that chooses signals myopically in every period $t' \geq t+1$. Let the resulting signal count vectors be $\tilde{d}(t')$, and define $\tilde{d}(t) = d(t)$. This deviation provides an upper bound on $W(t+1)$, given by

$$W(t+1) \leq (1-\delta)\sum_{t' \geq t+1} \delta^{t'-t-1} \cdot V(\tilde{d}(t')).$$

Since $W(t) = (1-\delta) \cdot V(d(t)) + \delta \cdot W(t+1)$, we have

$$\frac{1}{W(t+1)} - \frac{1}{W(t)} = \frac{(1-\delta) \cdot (V(d(t)) - W(t+1))}{W(t+1) \cdot ((1-\delta) \cdot V(d(t)) + \delta \cdot W(t+1))},$$

which is decreasing in $W(t+1)$ (holding $V(d(t))$ equal). Thus from the previous upper bound on $W(t+1)$, we obtain that

$$\frac{1}{W(t+1)} - \frac{1}{W(t)} \geq \frac{1}{\sum_{j=0}^{\infty}(1-\delta)\delta^j \cdot V(\tilde{d}(t+1+j))} - \frac{1}{\sum_{j=0}^{\infty}(1-\delta)\delta^j \cdot V(\tilde{d}(t+1+j))} \tag{14}$$

By the assumption that $C'QC \succeq Lc_jc_j'$ after t periods, we can apply (11) to deduce that for each $j \geq 0$,

$$\frac{1}{V(\tilde{d}(t+1+j))} - \frac{1}{V(\tilde{d}(t+j))} \geq \frac{L}{(L+1)\phi(\mathcal{S}^*)^2}.$$

Given this and (14), the desired result (13) follows from the technical lemma below (with $a = \frac{L}{(L+1)\phi(\mathcal{S}^*)^2}$, $x_j = V(\tilde{d}(t+1+j))$, $y_j = V(\tilde{d}(t+j))$ and $\beta_j = (1-\delta)\delta^j$):

Lemma 13. *Suppose a is a positive number. $\{x_j\}_{j=0}^{\infty}, \{y_j\}_{j=0}^{\infty}$ are two sequences of positive numbers such that $\frac{1}{x_j} \geq \frac{1}{y_j} + a$ for each j. Then for any sequence of positive numbers $\{\beta_j\}_{j=0}^{\infty}$ that sum to 1, it holds that*

$$\frac{1}{\sum_{j=0}^{\infty} \beta_j x_j} \geq \frac{1}{\sum_{j=0}^{\infty} \beta_j y_j} + a.$$

To see why this lemma holds, note that it is without loss to assume $\frac{1}{x_j} = \frac{1}{y_j} + a$ holds with equality. Then

$$1 - a \sum_j \beta_j x_j = \sum_j \beta_j (1 - a x_j) = \beta_j \frac{x_j}{y_j}$$

By the Cauchy-Schwarz inequality,

$$\sum_j \beta_j \frac{x_j}{y_j} \geq \frac{1}{\sum_j \beta_j \frac{y_j}{x_j}} = \frac{1}{\sum_j \beta_j (1 + a y_j)} = \frac{1}{1 + a \sum_j \beta_j y_j}.$$

So $1 - a \sum_j \beta_j x_j \geq \frac{1}{1 + a \sum_j \beta_j y_j}$, which is easily seen to be equivalent to $\frac{1}{\sum_j \beta_j x_j} \geq \frac{1}{\sum_j \beta_j y_j} + a$.

Hence Lemma 13 is proved, and so is Proposition 7. □

A.9.2 Efficiency as $\delta \to 1$

We now prove that for δ close to 1, the sampling strategy that maximizes δ-discounted payoff must eventually focus on the best complementary set $\mathcal{S}^*$. Recall that V^* is uniquely maximized at λ^*. Thus there exists positive η such that $V^*(\lambda) > (1 + \eta) V^*(\lambda^*)$ whenever λ puts zero frequency on at least one signal in $\mathcal{S}^*$.

Suppose for contradiction that sampling eventually focuses on a strongly complementary set $\mathcal{S}$ different from $\mathcal{S}^*$. Then at large periods t we must have $V(d(t)) > \frac{(1+\eta)\phi(\mathcal{S}^*)^2}{t}$, using the fact that V^* is the asymptotic version of V. As a result, there exists sufficiently large L_0 such that some signal in $\mathcal{S}^*$ is observed less than L_0 times under the optimal strategy for maximizing δ-discounted payoff.[39] Crucially, this L_0 can be chosen independently of δ. As a consequence, under the hypothesis of inefficient long-run outcome, $V(d(t)) > \frac{(1+\eta)\phi(\mathcal{S}^*)^2}{t}$ in fact holds for all $t > \underline{t}$ where $\underline{t}$ is also independent of δ.

Now we fix a positive integer $L > \frac{2}{\eta}$, and consider the following deviation strategy starting in period $\underline{t} + 1$:

1. In periods $\underline{t} + 1$ through $\underline{t} + Lk$, observe each signal in the best set $\mathcal{S}^*$ (of size k) exactly L times, in any order.

[39] Otherwise, $C'QC \succeq L_0 c_j c_j'$ holds at large t, implying a contradicting upper bound on $V(d(t))$ (see the argument in the previous subsection).

2. Starting in period $\underline{t} + Lk + 1$, sample myopically.

Let us study the posterior variance after period $\underline{t} + j$ under such a deviation. For $j \geq Lk + 1$, note that each signal $j \in \mathcal{S}^*$ has been observed at least L times before the period $\underline{t} + Lk + 1$. So $C'QC \succeq Lc_jc_j'$ holds, and we can deduce (similar to (12)) that the posterior variance is at most $(1 + \frac{1}{L}) \cdot \frac{\phi(\mathcal{S}^*)^2}{j - Lk}$. Since $\frac{1}{L} < \frac{\eta}{2}$, there exists $\underline{j}$ (depending on $\eta, \underline{t}, L, k$) such that the posterior variance after period $\underline{t} + j$ is at most $(1 + \eta/2)\frac{\phi(\mathcal{S}^*)^2}{\underline{t}+j}$ for $j > \underline{j}$. Thus the flow payoff *gain* in each such period is at least

$$\frac{\eta}{2} \cdot \frac{\phi(\mathcal{S}^*)^2}{\underline{t} + j}, \quad \forall j > \underline{j}$$

under this deviation strategy.

On the other hand, for $j \leq \underline{j}$ we can trivially bound the posterior variance from above by the prior variance V_0. This V_0 also serves as an upper bound on the flow payoff *loss* in these periods.

Combining both estimates, we find that the deviation strategy achieves payoff gain of at least

$$\delta^{\underline{t}} \cdot \left(\sum_{j > \underline{j}} \delta^{j-1} \cdot \frac{\eta}{2} \cdot \frac{\phi(\mathcal{S}^*)^2}{\underline{t} + j} - \sum_{j=1}^{\underline{j}} \delta^{j-1} \cdot V_0 \right).$$

Importantly, all other parameters in the above are constants independent of δ. As δ approaches 1, the sum $\sum_{j > \underline{j}} \frac{\delta^{j-1}}{\underline{t}+j}$ approaches a harmonic sum which diverges. Thus for all δ close to 1 the above display is strictly positive, suggesting that the constructed deviation is profitable. This contradiction completes the proof of Proposition 2 part (b).

A.10 Proofs for the Autocorrelated Model (Section 8)

A.10.1 Proof of Theorem 2

We work with the transformed model such that the signals in $\mathcal{S}$ become the first k transformed states $\tilde{\theta}_1, \ldots, \tilde{\theta}_k$. The payoff-relevant state becomes a certain linear combination $w_1\tilde{\theta}_1 + \cdots + \tilde{\theta}_k$ with positive weights $w_1, \ldots, w_k$. Choose M so that the innovations corresponding to the transformed states are independent from each other. In other words, $\tilde{M}$ (the transformed version of M) is given by $\text{diag}(\frac{x}{w_1}, \ldots, \frac{x}{w_k}, y_{k+1}, \ldots, y_K)$. Here x is a *small* positive number, while $y_{k+1}, \ldots, y_K$ are *large* positive numbers. We further choose $\Sigma^0 = M$, which is the stable belief without learning.

With these choices, it is clear that if all agents only sample from $\mathcal{S}$, society's beliefs about the transformed states remain independent at every period. Let v_i^{t-1} denote the prior variance about $\tilde{\theta}_i^t$ at the beginning of period t (before the signal acquisition in that period).

Then as long as agent t would continue to sample a signal $\tilde{\theta}_j + \mathcal{N}(0,1)$ in $\mathcal{S}$, these prior variances would evolve as follows: $v_i^0 = \frac{x}{w_i}$ for $1 \le i \le k$ and $v_i^0 = y_i$ for $i > k$. And for $t \ge 1$,

$$v_i^t = \begin{cases} \alpha \cdot v_i^{t-1} + (1-\alpha)\tilde{M}_{ii}, & \text{if } i \neq j; \\ \alpha \cdot \frac{v_i^{t-1}}{1+v_i^{t-1}} + (1-\alpha)\tilde{M}_{ii} & \text{if } i = j. \end{cases}$$

By induction, it is clear that $v_i^t \le \tilde{M}_{ii}$ holds for all pairs i, t, with equality for $i > k$. Thus at the beginning of each period t, assuming that all previous agents have sampled from $\mathcal{S}$, agent t's prior uncertainties about $\tilde{\theta}_1, \dots, \tilde{\theta}_k$ are small while his uncertainties about $\tilde{\theta}_{k+1}, \dots, \tilde{\theta}_K$ are large. As such, our previous proof for the existence of learning traps with persistent states carries over, and we deduce that agent t continues to observe from $\mathcal{S}$.

From this it is straightforward to show that each of the posterior variances v_i^t would admit a first-order approximation of $\frac{\sqrt{(1-\alpha)x \cdot \sum_{j=1}^k w_j}}{w_i}$ as $\alpha \to 1$ and $t \to \infty$. The posterior variance of $\sum_i w_i \tilde{\theta}_i^t$ is computed as $\sum_i w_i^2 \cdot v_i^t$, which is thus approximated as $\sqrt{(1-\alpha)x \cdot (\sum_{j=1}^k w_j)^3}$. This is exactly $\sqrt{(1-\alpha)\left(\frac{M_{11}}{\text{val}(\mathcal{S})}\right)}$ since $M_{11} = x \cdot \sum_{j=1}^k w_j$ and $\text{val}(\mathcal{S}) = \frac{1}{\phi(\mathcal{S})^2} = \frac{1}{(\sum_{j=1}^k w_j)^2}$. We thus deduce the payoff estimate in part (1) of the theorem.

A similar argument shows that myopically sampling from the best set $\mathcal{S}^*$ reduces long-run posterior variance to approximately $\sqrt{(1-\alpha)\left(\frac{M_{11}}{\text{val}(\mathcal{S}^*)}\right)}$, with $\text{val}(\mathcal{S}^*)$ replacing $\text{val}(\mathcal{S})$ in the denominator. This proves part (2) of Theorem 2.

A.10.2 Proof of Proposition 5

The environment in Example 4 is equivalent to one with three signals $\frac{1}{L}\omega$, $\frac{\omega+b}{2}$, $\frac{\omega-b}{2}$, each with standard Gaussian noise (just let $b = \omega + 2b_1$). We assume L is large, so that the best complementary set consists of the latter two signals.

For the autocorrelated model, we choose $M = \Sigma^0 = \text{diag}(x, x)$ with $x \ge L^2$. Then assuming that all previous agents have sampled the first signal, agent t's prior variance about b^t remains $x \ge L^2$. As such, he (and in fact each agent) continues to observe the first signal. In this case the prior variance v^t about ω^{t+1} evolves according to

$$v^t = \alpha \cdot \frac{L^2 \cdot v^{t-1}}{L^2 + v^{t-1}} + (1-\alpha)x.$$

It is not difficult to show that v^t must converge to the (positive) fixed point of the above equation. Let us in particular take $\alpha = 1 - \frac{1}{L^3}$ and $x = L^2$, then the long-run prior variance v solves $v = \frac{(L^2 - \frac{1}{L})v}{L^2+v} + \frac{1}{L}$. This yields exactly that $v = \sqrt{L}$. Hence long-run posterior variance is $\frac{L^2 \cdot v}{L^2+v} > \sqrt{L}/2$, which implies $\limsup_{\delta \to 1} U_\delta^M \le -\sqrt{L}/2$.

Let us turn to the optimal sampling strategy. Write $\tilde{\theta}_1 = \frac{\omega+b}{2}$ and $\tilde{\theta}_2 = \frac{\omega-b}{2}$. In this transformed model, $\tilde{M} = \tilde{\Sigma}^0 = \text{diag}(\frac{x}{2}, \frac{x}{2})$, and the payoff-relevant state is the sum of $\tilde{\theta}_1$ and $\tilde{\theta}_2$. Consider now a strategy that samples the latter two signals alternatively. Then the beliefs about $\tilde{\theta}_1$ and $\tilde{\theta}_2$ remain independent (as in $\tilde{M}$ and $\tilde{\Sigma}^0$), and their variances evolve as follows: $v_1^0 = v_2^0 = \frac{x}{2}$; in odd periods t

$$v_1^t = \alpha \cdot \frac{v_1^{t-1}}{1+v_1^{t-1}} + (1-\alpha)\frac{x}{2} \text{ and } v_2^t = \alpha \cdot v_2^{t-1} + (1-\alpha)\frac{x}{2},$$

and symmetrically for even t.

These imply that for odd t, v_1^t converges to v_1 and v_2^t converges to v_2 below (while for even t $v_1^t \to v_2$ and $v_2^t \to v_1$):

$$v_1 = \alpha \cdot \frac{\alpha v_1 + (1-\alpha)x/2}{1 + \alpha v_2 + (1-\alpha)x/2} + (1-\alpha)x/2;$$

$$v_2 = \alpha^2 \cdot \frac{v_2}{1+v_2} + (1-\alpha^2) \cdot \frac{x}{2}.$$

From the second equation, we obtain $(1-\alpha^2)(\frac{x}{2} - v_2) = \alpha^2 \cdot \frac{(v_2)^2}{1+v_2}$. With $\alpha = 1 - \frac{1}{L^3}$ and $x = L^2$, it follows that

$$v_2 = (1 + o(1))\frac{1}{\sqrt{L}}.$$

where $o(1)$ is a term that vanishes as $L \to \infty$. Thus we also have

$$v_1 = \alpha\frac{v_2}{1+v_2} + (1-\alpha)\frac{x}{2} = (1 + o(1))\frac{1}{\sqrt{L}}.$$

Hence under this alternating sampling strategy, long-run posterior variances about $\tilde{\theta}_1$ and $\tilde{\theta}_2$ are both bounded above by $\frac{2}{\sqrt{L}}$. Since $\omega = \tilde{\theta}_1 + \tilde{\theta}_2$, we conclude that $\liminf_{\delta \to 1} U_\delta^{SP} \geq -\frac{4}{\sqrt{L}}$. Choosing L large proves the proposition.

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B Other Results and Examples

B.1 Strengthening of Theorem 1 part (b)

Here we show the following result, which strengthens the restated Theorem 1 part (b) (see Appendix A.6). It says that any signal observed with zero long-run frequency must in fact be observed only finitely often.

Stronger Version of Theorem 1 part (b). *Suppose that the signals observed infinitely often span $\mathbb{R}^K$. Then $m_i(t) = \lambda_i^* \cdot t + O(1), \forall i$.*

The proof is divided into two subsections below.

B.1.1 Log Residual Term

Recall that we have previously shown $m_i(t) \sim \lambda_i^* \cdot t$. We can first improve the estimate of the residual term to $m_i(t) = \lambda_i^* \cdot t + O(\ln t)$. Indeed, Lemma 12 yields that for some constant L and every $t \geq L$,

$$V(m(t+1)) \leq V(m(t)) - \left(1 - \frac{L}{t}\right) \cdot \frac{V(m(t))^2}{\phi(\mathcal{S}^*)^2}. \tag{15}$$

This is because we may apply Lemma 12 with $M = \min_{j \in \mathcal{S}^*} m_j(t)$, which is at least $\frac{t}{L}$.

Let $g(t) = \frac{V(m(t))}{\phi(\mathcal{S}^*)^2}$. Then the above simplifies to

$$g(t+1) \leq g(t) - \left(1 - \frac{L}{t}\right) g(t)^2.$$

Inverting both sides, we have

$$\frac{1}{g(t+1)} \geq \frac{1}{g(t)} + \frac{1 - L/t}{1 - (1 - L/t)g(t)} \geq \frac{1}{g(t)} + 1 - \frac{L}{t}. \tag{16}$$

This enables us to deduce

$$\frac{1}{g(t)} \geq \frac{1}{g(L)} + \sum_{x=L}^{t-1} \left(1 - \frac{L}{x}\right) \geq t - O(\ln t).$$

Thus $g(t) \leq \frac{1}{t - O(\ln t)} \leq \frac{1}{t} + O(\frac{\ln t}{t^2})$. That is,

$$V(m(t)) \leq \frac{\phi(\mathcal{S}^*)^2}{t} + O\left(\frac{\ln t}{t^2}\right).$$

Since $t \cdot V(\lambda t)$ approaches $V^*(\lambda)$ at the rate of $\frac{1}{t}$, we have

$$V^*\left(\frac{m(t)}{t}\right) \leq t \cdot V(m(t)) + O\left(\frac{1}{t}\right) \leq \phi(\mathcal{S}^*)^2 + O\left(\frac{\ln t}{t}\right). \tag{17}$$

Suppose $\mathcal{S}^* = \{1, \dots, k\}$. Then the above estimate together with (10) implies $\sum_{j>k} \frac{m_j(t)}{t} = O(\frac{\ln t}{t})$. Hence $m_j(t) = O(\ln t)$ for each signal j outside of the best set.

Now we turn attention to those signals in the best set. If these were the only available signals, then the analysis in Liang, Mu and Syrgkanis (2017) gives $\partial_i V(m(t)) = -\left(\frac{\beta_i^{\mathcal{S}^*}}{m_i(t)}\right)^2$. In our current setting, signals $j > k$ affect this marginal value of signal i, but the influence is limited because $m_j(t) = O(\ln t)$. Specifically, we can show that

$$\partial_i V(m(t)) = -\left(\frac{\beta_i^{\mathcal{S}^*}}{m_i(t)}\right)^2 \cdot \left(1 + O\left(\frac{\ln t}{t}\right)\right).$$

This then implies $m_i(t) \leq \lambda_i^* \cdot t + O(\ln t)$.[40] Using $\sum_{i \leq k} m_i(t) = t - O(\ln t)$, we deduce that $m_i(t) \geq \lambda_i^* \cdot t - O(\ln t)$ must also hold. Hence $m_i(t) = \lambda_i^* \cdot t + O(\ln t)$ for each signal i.

B.1.2 Getting Rid of the Log

In order to remove the $\ln t$ residual term, we need a refined analysis. The reason we ended up with $\ln t$ is because we used (15) and (16) at *each* period t; the "$\frac{L}{t}$" term in those equations adds up to $\ln t$. In what follows, instead of quantifying the variance reduction in each period (as we did), we will lower-bound the variance reduction over multiple periods. This will lead to better estimates and enable us to prove $m_i(t) = \lambda_i^* \cdot t + O(1)$.

To give more detail, let $t_1 < t_2 < \dots$ denote the periods in which some signal $j > k$ is chosen. Since $m_j(t) = O(\ln t)$ for each such signal j, $t_l \geq 2^{\epsilon \cdot l}$ holds for some positive constant ϵ and each positive integer l. Continuing to let $g(t) = \frac{V(m(t))}{\phi(\mathcal{S}^*)^2}$, our goal is to estimate the difference between $\frac{1}{g(t_{l+1})}$ and $\frac{1}{g(t_l)}$.

Ignoring period t_{l+1} for the moment, we are interested in $\frac{\phi(\mathcal{S}^*)^2}{V(m(t_{l+1}-1))} - \frac{\phi(\mathcal{S}^*)^2}{V(m(t_l))}$, which is just the difference in the *precision* about ω when the division vector changes from $m(t_l)$ to $m(t_{l+1} - 1)$. From the proof of Lemma 11, we can estimate this difference if the change were along the direction λ^*:

$$\frac{\phi(\mathcal{S}^*)^2}{V(m(t_l) + \lambda^*(t_{l+1} - 1 - t_l))} - \frac{\phi(\mathcal{S}^*)^2}{V(m(t_l))} \geq t_{l+1} - 1 - t_l. \tag{18}$$

Now, the vector $m(t_{l+1} - 1)$ is not exactly equal to $m(t_l) + \lambda^*(t_{l+1} - 1 - t_l)$, so the above estimate is not directly applicable. However, by our definition of t_l and t_{l+1}, any difference

[40] Otherwise, consider $\tau + 1 \leq t$ to be the last period in which signal i was observed. Then $m_i(\tau)$ is larger than $\lambda_i^* \cdot \tau$ by several $\ln(\tau)$, while there exists some other signal $\hat{i}$ in the best set with $m_{\hat{i}}(\tau) < \lambda_i^* \cdot \tau$. But then $|\partial_i V(m(\tau))| < |\partial_{\hat{i}} V(m(\tau))|$, meaning that the agent in period $\tau + 1$ should not have chosen signal i.

between these vectors must be in the first k signals. In addition, the difference is bounded by $O(\ln t_{l+1})$ by what we have shown. This implies[41]

$$V(m(t_{l+1}-1)) - V(m(t_l)+\lambda^*(t_{l+1}-1-t_l)) = O\left(\frac{\ln^2 t_{l+1}}{t_{l+1}^3}\right).$$

Since $V(m(t_{l+1}-1))$ is on the oder of $\frac{1}{t_{l+1}}$, we thus have (if the constant L is large)

$$\frac{\phi(\mathcal{S}^*)^2}{V(m(t_{l+1}-1))} - \frac{\phi(\mathcal{S}^*)^2}{V(m(t_l)+\lambda^*(t_{l+1}-1-t_l))} \geq -\frac{L\ln^2 t_{l+1}}{t_{l+1}}. \tag{19}$$

(18) and (19) together imply

$$\frac{1}{g(t_{l+1}-1)} \geq \frac{1}{g(t_l)} + (t_{l+1}-1-t_l) - \frac{L\ln^2 t_{l+1}}{t_{l+1}}.$$

Finally, we can apply (16) to $t = t_{l+1}-1$. Altogether we deduce

$$\frac{1}{g(t_{l+1})} \geq \frac{1}{g(t_l)} + (t_{l+1}-t_l) - \frac{2L\ln^2 t_{l+1}}{t_{l+1}}.$$

Now observe that $\sum_l \frac{2L\ln^2 t_{l+1}}{t_{l+1}}$ converges (this is the sense in which our estimates here improve upon (16), where $\frac{L}{t}$ leads to a divergent sum). Thus we are able to conclude

$$\frac{1}{g(t_l)} \geq t_l - O(1), \quad \forall l.$$

In fact, this holds also at periods $t \neq t_l$. Therefore $V(m(t)) \leq \frac{\phi(\mathcal{S}^*)^2}{t} + O(\frac{1}{t^2})$, and

$$V^*\left(\frac{m(t)}{t}\right) \leq t \cdot V(m(t)) + O\left(\frac{1}{t}\right) \leq \phi\left(\mathcal{S}^*\right)^2 + O\left(\frac{1}{t}\right). \tag{20}$$

This equation (20) improves upon the previously-derived (17). Hence by (10) again, $m_j(t) = O(1)$ for each signal $j > k$. And once these signal counts are fixed, $m_i(t) = \lambda_i^* \cdot t + O(1)$ also holds for signals i in the best set, as already proved in Liang, Mu and Syrgkanis (2017). This completes the proof.

B.2 Example of a Learning Trap with Non-Normal Signals

The payoff-relevant state $\theta \in \{\theta_1, \theta_2\}$ is binary and agents have a uniform prior. There are three available information sources. The first, X_1, is described by the information structure

	θ_1	θ_2
s_1	p	$1-p$
s_2	$1-p$	p

[41] By the mean-value theorem, the difference can be written as $O(\ln t_{l+1})$ multiplied by a certain directional derivative. Since the coordinates of $m(t_{l+1}-1)$ and of $m(t_l)+\lambda^*(t_{l+1}-1-t_l)$ both sum to $t_{l+1}-1$, this directional derivative has a direction vector whose coordinates sum to zero. Combined with $\partial_i V(m(t)) = -(\frac{\phi(\mathcal{S}^*)^2}{t}) \cdot (1+O(\frac{\ln t}{t}))$ (which we showed before), this directional derivative has size $O(\frac{\ln t}{t^3})$.

with $p > 1/2$. Information sources 2 and 3 provide perfectly correlated signals (conditional on θ) taking values in $\{a, b\}$: In state θ_1, there is an equal probability that $X_2 = a$ and $X_3 = b$ or $X_2 = b$ and $X_3 = a$. In state θ_2, there is an equal probability that $X_2 = X_3 = a$ and $X_2 = X_3 = b$.

In this environment, every agent chooses to acquire the noisy signal X_1, even though one observation of each of X_2 and X_3 would perfectly reveal the state.[42]

B.3 Example Mentioned in Section 7.1

Suppose the available signals are

$$\begin{aligned}
X_1 &= 10x + \epsilon_1 \\
X_2 &= 10y + \epsilon_2 \\
X_3 &= 4x + 5y + 10b \\
X_4 &= 8x + 6y - 20b
\end{aligned}$$

where $\omega = x + y$ and b is a payoff-irrelevant unknown. Set the prior to be

$$(x, y, b)' \sim \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0.1 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0.039 \end{pmatrix}\right).$$

It can be computed that agents observe only the signals X_1 and X_2, although the set $\{X_3, X_4\}$ is optimal with $\phi(\{X_1, X_2\}) = 1/5 > 3/16 = \phi(\{X_3, X_4\})$. Thus, the set $\{X_1, X_2\}$ constitutes a learning trap for this problem. But if each signal choice were to produce ten independent realizations, agents starting from the above prior would observe only the signals X_3 and X_4. This breaks the learning trap.

B.4 Supplementary Material to Section 9

B.4.1 General Payoff Functions

We comment here on the possibilities for (and limitations to) generalizing Proposition 2 beyond the quadratic loss payoff function. As discussed in the main text, the property that one-dimensional normal signals are Blackwell-ordered implies that part (a) of the proposition extends to general payoff functions. In other words, the frequency vector λ^* always maximizes long-run precision per signal.

On the other hand, at least for *some* other "prediction problems," λ^* continues to be the optimal frequency vector for maximizing the patient discounted average payoff. In a

[42] We thank Andrew Postlewaite for this example.

prediction problem, every agent's payoff function $u(a, \omega)$ is the same and depends only on $|a - \omega|$. For example, we can show that part (b) of Proposition 2 holds for $u(a, \omega) = |a - \omega|^\gamma$ with any exponent $\gamma \in (0, 2]$.

Nonetheless, even restricting to prediction problems, that part of the proposition does *not* hold in general. For a counterexample, consider $u(a, \omega) = -\mathbf{1}_{\{|a-\omega|>1\}}$, which punishes the agent for any prediction that differs from the true state by more than 1.[43] Intuitively, the payoff gain from further information decreases sharply (indeed, exponentially) with the amount of information that has already been acquired. Thus, even with a forward-looking objective function, the range of future payoffs is limited and each agent cares mostly to maximize his own payoff. This results in an optimal sampling strategy that resembles myopic behavior, and differs from the rule that would maximize speed of learning.

The above counterexample illustrates the difficulty in estimating the value of information when working with an arbitrary payoff function. In order to make intertemporal payoff comparisons, we need to know how much payoff is gained/lost when the posterior variance is decreased/increased by a certain amount. This can be challenging in general, see Chade and Schlee (2002) for a related discussion.[44]

Finally, while it is more than necessary to assume that agents have the same payoff function, the truth of part (b) of Proposition 2 does require some restrictions on how the payoff functions differ. Otherwise, suppose for example that payoffs take the form $-\alpha_t(a_t - \omega)^2$, where α_t decreases exponentially fast. Then even with the δ-discounted objective, the Social Planner puts most of the weight on earlier agents, making it optimal to acquire signals myopically.

B.4.2 Low Altruism

Here we show that part (a) of Theorem 1 generalizes to agents who are not completely myopic, but are sufficiently impatient. That is, we will show that if signals $1, \ldots, k$ are strongly complementary, then there exist priors given which agents with low discount factor ρ always observe these signals in equilibrium.

We follow the construction in Appendix A.5. The added difficulty here is to show that if any agent ever chooses a signal $j > k$, the payoff loss in that period (relative to myopically choosing among the first k signals) is at least a constant fraction of possible payoff gains in future periods. Once this is proved, then for sufficiently small ρ such a deviation is not profitable.

Suppose that agents sample only from the first k signals in the first $t - 1$ periods, with

[43] We thank Alex Wolitzky for this example.

[44] This difficulty becomes more salient if we try to go beyond prediction problems: The value of information in that case will depend on signal realizations.

frequencies close to λ^*. Then, the posterior variances $V_{11}, \ldots, V_{kk}$ (which are also the prior for period t) are on the order of $\frac{1}{t}$. Thus, following the computation in Appendix A.5, we can show that for some positive constant ξ (independent of t), the variance reduction of ω by any signal $j > k$ is at least $\frac{\xi}{t^2}$ smaller than the variance reduction by signal 1. This is the amount of payoff loss in period t under a deviation to signal j.

Such a deviation could improve the posterior variance in future periods. But even for the best continuation strategy, the posterior variance in period $t+m$ could at most be reduced by $O(\frac{m}{t^2})$.[45] Thus if we choose ξ to be small enough, the payoff gain in each period $t+m$ is bounded above by $\frac{m}{\xi t^2}$. Note that for ρ sufficiently small,

$$-\frac{\xi}{t^2} + \sum_{m \geq 1} \rho^m \cdot \frac{m}{\xi t^2} < 0.$$

Hence the deviation is not profitable and the proof is complete.

B.4.3 Multiple Payoff-Relevant States

Here we consider an extension of our main model, where each agent chooses the signal that minimizes a weighted sum of posterior variances about some r payoff-relevant states. These states and their weights are the same across agents. As before, let $V(q_1, \ldots, q_N)$ denote this weighted posterior variance as a function of the signal counts. V^* is the normalized, asymptotic version of V.

We assume that V^* is uniquely minimized at some frequency vector λ^*. Part (a) of Proposition 2 extends and implies that λ^* maximizes speed of learning. Unlike the case of $r = 1$, this optimal frequency vector generally involves more than K signals if $r > 1$.[46] We are not aware of any simple method to characterize λ^*.

Nonetheless, We can generalize the notion of "complementary sets" as follows: A set of signals $\mathcal{S}$ is complementary if both of the following properties hold:

1. each of the r payoff-relevant states is spanned by $\mathcal{S}$;

2. the optimal frequency vector supported on $\mathcal{S}$ puts positive weight on *each* signal in $\mathcal{S}$.

Similarly, we say that a complementary set $\mathcal{S}$ is "strongly complementary" if it is best in its subspace: the optimal frequency vector supported on $\overline{\mathcal{S}}$ only puts positive weights on signals in $\mathcal{S}$. When $r = 1$, these definitions agree with our main model.

By this definition, the existence of learning traps readily extends: For suitable prior beliefs, the marginal value of each signal in $\mathcal{S}$ persistently exceeds the marginal value of each

[45] This is because over m periods, the increase in the precision matrix is at most linear in m.

[46] A theorem of Chaloner (1984) shows that λ^* is supported on at most $\frac{r(2K+1-r)}{2}$ signals.

signal in $\overline{\mathcal{S}} - \mathcal{S}$. Since the marginal values of the remaining signals (those outside of the subspace) can be made very low by imposing large prior uncertainty on the relevant confounding terms, we deduce that society exclusively observes from the strongly complementary set $\mathcal{S}$.

We mention that part (b) of Theorem 1 also generalizes. For that we need a different proof, since there is no obvious analogue of Lemma 11 (and thus of Lemma 12) when $r > 1$. Instead, we prove the restated Theorem 1 part (b) in Appendix A.6 as follows: When society infinitely samples a set that spans $\mathbb{R}^K$, the marginal value of each signal j can be approximated by its asymptotic version:

$$\partial_i V(q_1, \dots, q_N) \sim \frac{1}{t^2} \cdot \partial_i V^*(\frac{q_1}{t}, \dots, \frac{q_N}{t}).$$

Together with Lemma 10, this shows that the myopic signal choice j in any sufficient late period must *almost* minimize the partial derivative of V^*, in the following sense:

Lemma 14. *For any $\epsilon > 0$, there exists sufficiently large $t(\epsilon)$ such that if signal j is observed in any period $t+1$ later than $t(\epsilon)$, then*

$$\partial_j V^* \left(\frac{m(t)}{t}\right) \leq (1-\epsilon) \min_{1 \leq i \leq N} \partial_i V^* \left(\frac{m(t)}{t}\right).$$

Consider society's frequency vectors $\lambda(t) = \frac{m(t)}{t} \in \Delta^{N-1}$. Then they evolve according to

$$\lambda(t+1) = \frac{t}{t+1}\lambda(t) + \frac{1}{t+1}e_j.$$

whenever j is the signal choice in period $t+1$. So the frequencies $\lambda(t)$ move in the direction of e_j, which is the direction where V^* decreases almost the fastest. This suggests that the evolution of $\lambda(t)$ over time resembles the gradient descent dynamics. As such, we can expect that the value of $V^*(\lambda(t))$ roughly decreases over time, and that eventually $\lambda(t)$ approaches $\lambda^* = \operatorname{argmin} V^*$.

To formalize this argument, we have (for fixed $\epsilon > 0$ and sufficiently large t)

$$\begin{aligned} V^*(\lambda(t+1)) &= V^*\left(\frac{t}{t+1}\lambda(t) + \frac{1}{t+1}e_j\right) \\ &= V^*\left(\frac{t}{t+1}\lambda(t)\right) + \frac{1}{t+1} \cdot \partial_j V^*\left(\frac{t}{t+1}\lambda(t)\right) + O\left(\frac{1}{(t+1)^2} \cdot \partial_{jj} V^*\left(\frac{t}{t+1}\lambda(t)\right)\right) \\ &\leq V^*\left(\frac{t}{t+1}\lambda(t)\right) + \frac{1-\epsilon}{t+1} \cdot \partial_j V^*\left(\frac{t}{t+1}\lambda(t)\right) \\ &= \frac{t+1}{t} \cdot V^*(\lambda(t)) + \frac{(1-\epsilon)(t+1)}{t^2} \cdot \partial_j V^*(\lambda(t)) \\ &\leq V^*(\lambda(t)) + \frac{1}{t} \cdot V^*(\lambda(t)) + \frac{1-2\epsilon}{t} \cdot \min_{1 \leq i \leq N} \partial_i V^*(\lambda(t)). \end{aligned} \tag{21}$$

The first inequality uses Lemma 9, the next equality uses the homogeneity of V^*, and the last inequality uses Lemma 14.

Write $\lambda = \lambda(t)$ for short. Note that V^* is differentiable at λ, since $\lambda_i(t) > 0$ for a set of signals that spans the entire space. Thus the convexity of V^* yields

$$V^*(\lambda^*) \geq V^*(\lambda) + \sum_{i=1}^{N} (\lambda_i^* - \lambda_i) \cdot \partial_i V^*(\lambda).$$

The homogeneity of V^* implies $\sum_{i=1}^{N} \lambda_i \cdot \partial_i V^*(\lambda) = -V^*(\lambda)$. This enables us to rewrite the preceding inequality as

$$\sum_{i=1}^{N} \lambda_i^* \cdot \partial_i V^*(\lambda) \leq V^*(\lambda^*) - 2V^*(\lambda).$$

Thus, in particular,

$$\min_{1 \leq i \leq N} \partial_i V^*(\lambda(t)) \leq V^*(\lambda^*) - 2V^*(\lambda). \tag{22}$$

Combining (21) and (22), we have for all large t:

$$V^*(\lambda(t+1)) \leq V^*(\lambda(t)) + \frac{1}{t} \cdot [(1 - 2\epsilon) \cdot V^*(\lambda^*) - (1 - 4\epsilon) \cdot V^*(\lambda(t))]. \tag{23}$$

Now, suppose (for contradiction) that $V^*(\lambda(t)) > (1 + 4\epsilon) \cdot V^*(\lambda^*)$ holds for all large t. Then (23) would imply $V^*(\lambda(t+1)) \leq V^*(\lambda(t)) - \frac{\epsilon \cdot V^*(\lambda^*)}{t}$. But since the harmonic series diverges, $V^*(\lambda(t))$ would eventually decrease to be negative, which is impossible. Thus

$$V^*(\lambda(t)) \leq (1 + 4\epsilon) \cdot V^*(\lambda^*)$$

must hold for *some* large t. By (23), the same is true at all future periods. But since ϵ is arbitrary, the above inequality proves that $V^*(\lambda(t)) \to V^*(\lambda^*)$. Hence $\lambda(t) \to \lambda^*$, completing the proof of Theorem 1 for multiple payoff-relevant states.

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