Problem Set 3

Due: in class, March 27th

1 Gambler's and hot hand fallacy

1. Please describe what the phenomena of gambler's fallacy and hot hand fallacy are

2. Suppose an analyst wants to forecast if a coin will be heads or tails: the coin tosses are *iid* with the probability of tails being θ. Suppose she believes in the law of small numbers as follows. She believes, as in Rabin (2002),[1] that the coin behaves as an urn containing N balls, from which the balls are extracted without replacement, where the initial proportion of tails is θ, but the urn is replaced every odd period with a new one with N elements and the same fraction θ. That is

$$Pr(Tails_t \,|\, Tails_{t-1}) = Pr(Tails_1) = \theta \,\text{if}\, t \,\text{is odd}$$

while

$$Pr(Tails_t \,|\, Tails_{t-1}) = \frac{N\theta - 1}{N-1} \text{ if } t \text{ is even}$$

Suppose the analyst knows $\theta = \frac{1}{2}$.

   (a) When does she exhibit the gambler's fallacy?

   (b) What is the analyst's assessment of the probability of three heads and one tails (not in a particular order)? What is the true probability? Comment.

   (c) What is the analyst's assessment of the probability of two heads and two tails (not in a particular order)? What is the true probability? Comment.

3. Suppose the analyst is uncertain on whether the coin is fair or not. At time 0 he believes that, with $\frac{1}{2}$ probability,θ is equal to $\frac{1}{4}$ and, with $\frac{1}{2}$ probability, $\theta = \frac{3}{4}$. When she observes a signal, ($Tails$ or $Heads$), she updates her prior using bayes' rule. Note that the likelihood is $Pr(Tails_t | \, \theta) = \theta$ when t is odd, and $Pr(Tails_t | \, \theta) = \frac{N\theta - 1}{N}$ when T is even

[1] Rabin, M., 2002. Inference by believers in the law of small numbers. The Quarterly Journal of Economics, 117(3), pp.775-816.

(a) Compare the analyst's assessments $Pr(Tails_2 | Tails_1)$, $Pr(\theta = \frac{3}{4} | Tails_2, Tails_1)$ and $Pr(Tails_3 | Tails_2, Tails_1)$ with those of an analyst that does not believe in the law of small numbers, i.e. understands that the coin tosses are i.i.d.

(b) By interpreting point (a), in what sense can we say that the analyst exhibits both the hot hand and the gambler's fallacy? Explain what is the link between the two in this model.

4. Suppose now the analyst wants to forecast the return of a stock. The returns at time 1, 2, 3 are a constant plus a random *iid* shock[2]

$$r_t = \mu + \epsilon_t \qquad t = 1, 2, 3$$

$$\begin{array}{ll} \epsilon_t \perp \epsilon_s & \forall\, s \neq t \\ \mathbb{E}[\epsilon_t] = 0 & t = 1, 2.3 \end{array}$$

At each time t she observes r_t. Moreover, assume she does not believe $\epsilon_t \perp \epsilon_s$: instead she thinks that

$$\epsilon_t = \omega_t - \sum_{j=1}^{t} \delta^j \epsilon_{t-j}$$

where δ is a parameter in $[0, 1]$, $\omega_t \perp \omega_s$for any $s \neq t$ and $\mathbb{E}[\omega_t] = 0$

(a) Write r_t as a function of μ, ω_t and $\{r_j\}_{j=1}^{t-1}$

(b) Assume the analyst knows $\mu = 0$. Suppose at time 1 she observes $r_1 > 0$ what is the analyst's forecast $\mathbb{E}[r_2 \,|\, r_1 > 0]$? What would an analyst that knows $\epsilon_t \perp \epsilon_s$ forecast? In what sense the biased analyst exhibits the gambler's fallacy?

5. **(Optional: extra credit)** Suppose the analyst does not know μ. At time 0 she has a prior distribution over μ

$$\mu \sim \mathcal{N}(0, \sigma_\mu^2)$$

where $\sigma_\mu^2 > 0$. She believes

$$\omega_t \sim_{i.i.d.} \mathcal{N}(0, \sigma^2)$$

with $\sigma^2 > 0$

(a) Find the analyst's forecasts $\mathbb{E}[s_2 | \, s_1]$, $\mathbb{E}[s_3 | \, s_1, s_2]$.

(b) Suppose $s_1, s_2 > 0$, and compare $\mathbb{E}[\mu | \, s_1]$ and $\mathbb{E}[\mu | \, s_1, s_2]$ to the case where the analyst knows $\epsilon_t \perp \epsilon_s$ (that is when$\delta = 0$). Comment. Look at $\mathbb{E}[s_3 | \, s_1, s_2]$: can the analyst exhibit something we could interpret as a hot hand fallacy? Comment.

[2]when in this exercise I write $\perp$ I mean "independent of"

2 Overreaction and underreaction in diagnostic expectations

Suppose a bayesian agent A and a representativeness biased agent B forecast a discrete random variable X. There are three periods: 0, 1, 2. X is realized at time 2. The two agents receive information set I_0 at time 0 and I_1 at time 1. Let us call $p_{X \mid I_t}$ the distribution of X that A believes in, conditional on information I_t, and $p^{\theta}_{X \mid I_t}$the distribution of X that B believes in. Hence, let us call the expectation of agent A conditional on information I_t is

$$\mathbb{E}\left[X \mid I_t\right] := \sum_{x \in \mathbb{R}} x p_{X \mid I_t}(x)$$

while expectation of agent$B$ conditional on information I_t is

$$\mathbb{E}_{\theta}\left[X \mid I_t\right] := \sum_{x \in \mathbb{R}} x p^{\theta}_{X \mid I_t}(x)$$

- At time 0, given I_0, the agents agree that there is an objective distribution of X , $p(x)$, that is

$$p_{X \mid I_0}(x) = p^{\theta}_{X \mid I_0}(x) = p(x) \tag{1}$$

  $\forall x \in \mathbb{R}$

- At time 1, agents receive the same information $I_1$and update their distributions differently. Agent A updates using bayes rule and formulates $p_{X \mid I_0}$. Agent B instead, is biased by representativeness and updates his distribution as follows

$$p^{\theta}_{X \mid I_1}(x) = \frac{p_{X \mid I_1}(x)\left(\frac{p_{X \mid I_1}(x)}{p_{X \mid I_0}(x)}\right)^{\theta}}{\sum_{z \in \mathbb{R}} p_{X \mid I_1}(z)\left(\frac{p_{X \mid I_1}(z)}{p_{X \mid I_0}(z)}\right)^{\theta}} \tag{2}$$

  that, given (1) is equal to

$$\frac{p_{X \mid I_1}(x)\left(\frac{p_{X \mid I_1}(x)}{p(x)}\right)^{\theta}}{\sum_{z \in \mathbb{R}} p_{X \mid I_1}(z)\left(\frac{p_{X \mid I_1}(z)}{p(z)}\right)^{\theta}}$$

1. Interpret equation (2) in light of what you have learned about stereotypes[3] (what are G and $-G$, the groups conditioning on which a representativeness biased thinker evaluates the representativeness of a trait?).Which realizations of X receive more weight than in the bayesian case?

[3]Stereotypes are defined in Bordalo, P., Coffman, K., Gennaioli, N. and Shleifer, A., 2016. Stereotypes. The Quarterly Journal of Economics, 131(4), pp.1753-1794.

2. Suppose that $\frac{p_{X \mid I_1}(x)}{p_{X \mid I_0}(x)}$ is strictly increasing in x, that is, I_1 brings larger positive surprise on large realizations of X. Show that agent B overreacts to information I_1, that is

$$\mathbb{E}_\theta [X \mid I_1] > \mathbb{E} [X \mid I_1] > \mathbb{E} [X \mid I_0]$$

(If you cannot: it suffices to show it for the cases when X takes two and three values). Give an interpretation.[4]

3. Now suppose that at period 0, the agents contemplate 5 possible states

realizations	x_1	x_2	x_3	x_1	x_2
states	ω_1	ω_2	ω_3	ω_4	ω_5
probabilities	$\bar{p}$	$\tilde{p}$	$\hat{p}$	$\bar{p}$	$\tilde{p}$

where

$$2\bar{p} + 2\tilde{p} + \hat{p} = 1$$

and

$$x_3 > x_2 > x_1.$$

Let us assume that information at time 0 is

$$I_0 = \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5\}.$$

Suppose that at time 1 agents receive information that they are either in state ω_4 or ω_5. That is

$$I_1 = \{\omega_4, \omega_5\}.$$

(a) Is it good news or bad news according to A and B?[5]

(b) When does B overreact relative to A? When does he underreact? [6]Give an interpretation.

---

[4]Note that, conversely, if $\frac{p_{X \mid I_1}(x)}{p_{X \mid I_0}(x)}$ is strictly decreasing in x, $\mathbb{E}_\theta [X \mid I_1] < \mathbb{E} [X \mid I_1] < \mathbb{E} [X \mid I_0]$

[5]That is, how do $\mathbb{E}_\theta [X \mid I_1]$ and $\mathbb{E} [X \mid I_1]$relate to time 0 expectation $\mathbb{E} [X \mid I_0]$?

[6]Formally overreaction is when either

$$\mathbb{E}_\theta [X \mid I_1] > \mathbb{E} [X \mid I_1] > \mathbb{E} [X \mid I_0]$$

or

$$\mathbb{E}_\theta [X \mid I_1] < \mathbb{E} [X \mid I_1] < \mathbb{E} [X \mid I_0]$$

. Underreaction is when either

$$\mathbb{E} [X \mid I_1] > \mathbb{E}_\theta [X \mid I_1] > \mathbb{E} [X \mid I_0]$$

or

$$\mathbb{E} [X \mid I_1] < \mathbb{E}_\theta [X \mid I_1] < \mathbb{E} [X \mid I_0]$$

4. Now suppose there are two binary random variables, X and Y and three states

realizations			
X	0	1	1
Y	0	1	0
states	ω_1	ω_2	ω_3
probabilities	p	p	$1-2p$

   Assume time 0 information is

$$I_0 = \{\omega_1, \omega_2, \omega_3\}.$$

   Suppose at time 1 agents receive information

$$I_1 = \{\omega_1, \omega_2\}.$$

   (a) Compare the $\mathbb{E}_\theta [X \mid I_1]$ and $\mathbb{E}_\theta [Y \mid I_1]$: is there anything surprising, in light of how would the standard bayesian agent predict? Interpret.

   (b) Assume uncertainty is resolved at time 3. Suppose at time 2 no new information comes, relative to $t = 1$. That is

$$I_2 = I_1 = \{\omega_1, \omega_2\}$$

   What are $\mathbb{E}_\theta [X \mid I_2]$ and $\mathbb{E}_\theta [Y \mid I_2]$? How do they relate to $\mathbb{E}_\theta [X \mid I_1]$ and $\mathbb{E}_\theta [Y \mid I_1]$? Provide an intuition.

3 True/False/Uncertain

1. In the paper "Decision making under the gambler's fallacy: Evidence from asylum judges, loan officers, and baseball umpires." by Chen, D.L., Moskowitz, T.J. and Shue, K., (QJE 2016), the three main facts documented are jointly consistent with the following explanation: the decision makers believe that there is only a given fraction of decisions that they should take in one direction.

2. Anchoring implies that people rely more on personal experiences of a very limited sample rather than on public information from a much larger sample.

3. In the model of stereotypes, the type that has the largest frequency within a group G will also be the most representative of that group.