Distortions in Production Networks

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Abstract

How does an economy’s production structure determine its macroeconomic response to sectoral distortions? We study a static, multi-sector framework in which production is organized in an input-output network and production decisions are distorted. Sectoral distortions manifest at the aggregate level via two channels: total factor productivity (TFP) and the labor wedge. We show that near efficiency, distortions have zero first-order effects on TFP, non-zero first-order effects on the labor wedge, and that a sufficient statistic for the latter are the Domar weights. We thereby provide a Hulten-like theorem for the aggregate effects of sectoral distortions. A quantitative application of the model to the 2008-09 Financial Crisis suggests that the U.S. input-output structure amplified financial distortions by roughly a factor of two during the crisis.

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1 Introduction

A downward glance from an airplane window can often attest to the incredible complexity behind modern economic production. Warehouses, silos, and freight loading zones adjoin manufacturing and industrial plants. A web of intersecting highways, railroads, and waterways carry shipments to and from individual producers. From the vast stretches of farmland and oil-rich shale fields, to the service and retail centers of large cities, millions of independent production nodes compose the intricate, yet harmonious system that is today’s economy.

This system, however, is not free from disturbances. At the micro level, firms face a changing array of frictions—e.g. taxation, financial constraints, and market power—all of which distort individual production choices away from efficient levels. In a complex network of intermediate good trade, these choices affect not only the firms themselves but also their trading partners, their trading partners’ partners, and so on. The aggregate effects of such frictions thereby depend on the underlying production network architecture.

In this paper we examine how an economy’s production structure determines its macroeconomic response to micro-level distortions.

The general framework. We study a relatively standard multi-sector, general equilibrium, static model of intermediate good trade. Firms operate constant returns-to-scale technologies that combine labor and intermediate goods to produce output. Sectoral output may be consumed by the household or used as intermediate goods; the matrix of cross-sector input requirements constitutes the economy’s input-output network. Firms are subject to sectoral productivity shocks, and a representative household consumes and supplies labor elastically.

Within this framework, we allow sectoral production choices to be distorted. Distortions are captured by wedges between firms’ prices and marginal costs. While we remain agnostic regarding the microeconomic origin of these distortions, we show how they may be rationalized by tax rates, mark-ups, working capital constraints, and the like.

We begin our analysis by showing that the equilibrium allocation in this economy can be mapped to the equilibrium of the static, prototypical, representative firm, representative household economy à la Chari, Kehoe, and McGrattan (2007). The equilibrium of the benchmark prototype economy is characterized by two equations: a constant returns-to-scale aggregate production function and an intratemporal optimality condition.

The aggregate production function admits an “efficiency wedge,” or total factor productivity (TFP) component, while the intratemporal condition admits a wedge between the marginal rate of substitution between consumption and labor and their marginal rate of transformation. Fluctuations in the latter, known as the “labor wedge,” or equivalently the aggregate markup, have been documented to account for a significant portion of macroeconomic variation at business cycle frequency (see e.g. Hall, 1997; Rotemberg and Woodford, 1999; Chari, Kehoe, and McGrattan, 2007).
McGrattan, 2007; Shimer, 2009; Karabarbounis, 2014)

The prototype economy is a “macro” representation of the economy; the efficiency wedge and the labor wedge are equilibrium objects that depend on the economy’s underlying shocks and primitives: preferences, technologies, and in particular the input-output structure. The primary focus of this paper is to characterize the macroeconomic effects of “micro” sectoral distortions via these two aggregate channels—the efficiency wedge and the labor wedge—with an emphasis on the latter.

Results. We begin by restating a well-known result attributed to Hulten (1978). Hulten’s theorem states that in an efficient economy, the Domar weights—defined as sales as a share of GDP—are sufficient statistics for the first-order effects of sectoral productivities on TFP. This theorem implies that to a first-order, knowledge of the underlying intricacies and microeconomic “details” governing the structure of an economy—that is, preferences, technologies, and input-output structures—is unnecessary for inferring the macroeconomic response to sectoral productivities. Sales shares are easily observable and suffice.

We then move on to the macroeconomic effects of sectoral distortions. Unlike productivity shocks, distortions do not actively create or destroy resources; instead, they simply drive wedges between firms’ prices and marginal costs. By moving production decisions away from their optima, distortions result in an inefficient reallocation of resources across the input-output network, thereby lowering Diamond and Mirrlees (1971) productive efficiency.

Our main result establishes that starting from efficiency, distortions have zero first-order effects on TFP but non-zero first-order effects on the labor wedge. Furthermore, a sufficient statistic for the latter—the first-order effects of sectoral distortions on the labor wedge—are the Domar weights.

The first part should come as no surprise. Diamond and Mirrlees (1971) productive efficiency is maximized when distortions are neutralized. Starting from this maximum, small distortions have zero first-order effects on TFP. Regardless of the network structure, misallocation generates a fall in TFP through second and higher-order terms alone.

Our contribution is the second part. While sectoral distortions generate no first-order loss in productive efficiency, they do produce first-order effects on the labor wedge. As distortions drive wedges between sectoral prices and marginal costs, the effects of these wedges compound as firms buy and sell from one another within the network of intermediate good trade. At the aggregate level, the price of the final consumption good is distorted away from its marginal cost of production, and a labor wedge arises.

However, to a first-order near efficiency, knowledge of the microeconomic details of the underlying economy—preferences, technologies, and the input-output structure—is again unnecessary insofar as sales shares are observable. Domar weights serve as sufficient statistics
for the first-order labor wedge response to sectoral distortions. We thereby provide a Hulten-like theorem for the labor wedge.

**Results for Cobb-Douglas economies.** Domar weights are equilibrium objects—they do not provide a structural, economic interpretation for how they are determined by the economy’s primitives: technologies, preferences, and the input-output structure. We thus proceed to restrict attention to economies with Cobb-Douglas preferences and technologies and, within this class, provide such an economic interpretation.

In the class of Cobb-Douglas economies, we show that near efficiency, the first-order effects of sectoral distortions on the labor wedge are succinctly characterized by a familiar vector of constants indicating each sectors’ network “centrality.” This vector, at times called the “influence vector,” has been shown to determine the elasticity of TFP to sectoral productivities in Cobb-Douglas economies (Burress, 1994; Acemoglu et al., 2012). The influence vector is a simple transformation of the Leontief inverse matrix. The Leontief inverse matrix captures the entire infinite sequence of all higher-order network effects; its mathematical structure is closely related to the concept of Bonacich centrality, a measure of “importance” within a network.

Within the class of Cobb-Douglas economies, we find that network centrality, as captured by the Leontief inverse matrix, is the primary determinant of the labor wedge response to sectoral distortions. Thus, distortions in sectors with high centrality—sectors with more direct and/or indirect downstream customers—have greater effects on the labor wedge than distortions in sectors with low centrality.

**Bottlenecks.** While the results for small distortions may be helpful for intuition, they hold only up to a first-order around efficiency. To the extent that the economy is away from efficiency, Domar weights and network “centralities” are no more sufficient. We thereby consider a special case, a “supply chain” economy, and use various limits of this economy to gain a better understanding for the aggregate effects of distortions away from efficiency.

These limits isolate a particular lesson. Namely, in any economy there exist multiple paths by which the primary factor, labor, is transformed into final consumption. There may, however, exist critical “bottleneck” sectors through which most or all of these paths must pass. We find that a distortion in a bottleneck sector can have a disproportionately large effect on the labor wedge, but at the same time contribute very little to TFP. This is because a bottleneck distortion drives a wedge into the price path of most commodities yet generates almost no misallocation. Finally, we show how this lesson generalizes to the larger class of Cobb-Douglas economies.

**Quantitative illustration.** In the final section of the paper we study a particular quantitative application. We calibrate the production network to the US input-output tables provided by the
Bureau of Economic Analysis (BEA) and evaluate the extent to which the U.S. network structure may have amplified within-period financial frictions during the 2008-2009 Financial Crisis. We abstract from productivity shocks and focus exclusively on distortions driven by working capital constraints. In particular, we use panel data on sectoral credit spreads constructed and provided by Gilchrist and Zakrajišek (2012) as our proxy for financial distortions. We feed the “GZ distortion” series into our calibrated network model and simulate its aggregate response.

The calibrated model with GZ distortions produces a fall in the labor wedge that is an order of magnitude larger than the model-implied fall in TFP. This quantitative result fits squarely with the theory: sectoral distortions have first-order effects on the labor wedge, but only second and higher-order effects on the efficiency wedge. Given that we abstract entirely from productivity shocks in our quantitative analysis, it is no surprise that the GZ distortions alone cannot result in large fluctuations in TFP.

The primary goal of our analysis is to quantify the role played by the U.S. input-output network in amplifying sectoral financial frictions during the crisis. We construct a measure we call the “labor wedge network multiplier” defined as the ratio between the percentage drop in the labor wedge generated by the calibrated network economy and that generated by a counterfactual, but otherwise equivalent, economy without input-output linkages. Our calibration produces a labor wedge network multiplier of roughly 2, suggesting that the U.S. input-output network generates considerable amplification of financial distortions.

Finally, we show how a certain back-of-the-envelope calculation helps one interpret this number. Our theoretical results indicate that to a first-order near efficiency, a sufficient statistic for the labor wedge network multiplier is the sum of the Domar weights. The sum of the Domar weights is itself equal to total sales over GDP; in our data, total sales over GDP is roughly 2.

**Related literature.** Our paper contributes to the broad literature mapping “micro” shocks to macroeconomic fluctuations.

Following the seminal work of Gabaix (2011), a large literature studies the aggregate effects of “granular” productivity shocks (see e.g. Foerster et al., 2011; Acemoglu et al., 2012; Carvalho and Gabaix, 2013; di Giovanni et al., 2014, 2018, among others). Gabaix (2011) shows that in an efficient economy, idiosyncratic firm-level productivity shocks can become macroeconomic fluctuations when the firm size distribution is sufficiently fat-tailed. One implication of our work is that the Gabaix (2011) result can in theory be extended to the first-order labor wedge effects of “granular” distortions. That is, near efficiency, idiosyncratic shocks to firm-level distortions, e.g. firm-level markups, can result in first-order labor wedge movement as long as the firm size distribution is fat-tailed.

An adjacent strand of the literature builds on the canonical multi-sector model of Long and Plosser (1983) and studies the aggregate effects of sectoral productivity shocks in input-output
We likewise adopt the Long and Plosser (1983) input-output economy as our starting point, but we instead shift focus toward the aggregate effects of sectoral distortions.

Our paper is thereby most closely related to the literature on the transmission of distortions in economies with intermediate good trade. Previous work studies the effects of distortions within specific input-output structures; see Basu (1995); Ciccone (2002); Yi (2003); Jones (2011); Asker et al. (2014). Our paper, along with work by Jones (2013), Baqae and Farhi (2020), and Liu (2019), characterize the macroeconomic effects of distortions in multi-sector economies with generic input-output structures.

Jones (2013) examines aggregate misallocation due to micro distortions in a Cobb-Douglas input-output economy. Baqae and Farhi (2020) study an economy with general constant returns-to-scale technologies and characterize the first-order effects of productivities and distortions on TFP both at and away from efficiency; in so doing, Baqae and Farhi (2020) generalize Hulten (1978) to equilibrium allocations away from the efficient one. Finally, Liu (2019) considers the aggregate efficiency implications of distortions whose payments are discarded. He studies the desirability of industrial policy: whether a central government should subsidize production in select sectors subject to this class of shocks.

The primary difference between our model and those of Jones (2013), Baqae and Farhi (2020), and Liu (2019) is the treatment of primary factors. While the supply of primary factors in those papers are exogenously fixed, we allow for the supply of labor—our primary factor of production—to be endogenously determined. This modeling choice has the following implication. With primary factors exogenously fixed, the distortions in Jones (2013), Baqae and Farhi (2020), and Liu (2019) manifest solely in aggregate efficiency, or TFP.

In contrast, we allow for the primary factor in our model, labor, to be supplied elastically by the household. Distortions thereby manifest in our context not only as fluctuations in TFP but also as aggregate movements in the labor wedge. The primary focus of our paper is the latter channel. Our characterization of the labor wedge effects of sectoral distortions within a generic input-output economy are, to the best of our knowledge, novel contributions of this paper.

**Layout.** This paper is organized as follows. Section 2 introduces the general framework. Section 3 defines and characterizes equilibrium allocations. Section 4 presents our main results on how sectoral distortions manifest at the macro level. Section 5 presents a simple and tractable special case. Section 6 provides a quantitative illustration of the model applied to the

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1See Carvalho (2014) and Carvalho and Tahbaz-Salehi (2019) for surveys of this literature.
2See also Baqae and Farhi (2019) for a characterization of the second-order TFP effects of productivity shocks in efficient economies.
3Jones (2013) assumes a fixed, exogenous aggregate supply of two primary factors: physical and human capital. Liu (2019) assumes a fixed, exogenous supply of one primary factor resembling labor. Baqae and Farhi (2020) allow for any finite number of primary factors, but all in exogenous fixed supply.
2008-2009 Financial Crisis. Section 7 concludes. The appendix contains all proofs of the results for the general economy. Proofs of the results for the Cobb-Douglas economy and all supporting material for the quantitative application can be found in the online appendix.

2 The Environment

The economy is static and populated by a representative household and \( N \) production sectors, indexed by \( i \in I = \{1, \ldots, N\} \). Each sector consists of two types of firms: (i) a unit mass of monopolistically-competitive firms, indexed by \( k \in [0, 1] \), producing differentiated goods, and (ii) a perfectly-competitive producer whose sole purpose is to aggregate the industry's differentiated goods into a uniform sectoral good. The output of each industry can either be consumed by the household or used as an intermediate input to production. Finally there is a government which levies sector-specific revenue taxes or subsidies and rebates part or all of the tax revenue back to the household.

Production. Technology is identical within sectors but heterogeneous across sectors. The production of firm \( k \) in sector \( i \) is given by

\[
y_{i,k} = z_i F_i(\ell_{i,k}, x_{i,k}),
\]

where \( y_{i,k} \) is the firm's output, \( \ell_{i,k} \) its labor, \( x_{i,k} \) a composite of the firm's intermediate inputs, and \( z_i \) is a sector-specific productivity shock. The firm's composite of intermediate goods is given by

\[
x_{i,k} \equiv G_i(x_{i,k}),
\]

where \( x_{i,k} \equiv (x_{i1,k}, \ldots, x_{iN,k})' \) denotes the vector of the firm's intermediate good purchases, and \( x_{ij,k} \) is the amount it purchases of the sectoral commodity \( j \).

We assume that all technologies, \( F_i : \mathbb{R}_+^2 \to \mathbb{R}_+ \) and \( G_i : \mathbb{R}_+^N \to \mathbb{R}_+ \) for all \( i \in I \), are Neoclassical: they are homogenous of degree 1 (exhibit constant returns-to-scale), are twice continuously-differentiable in all arguments, have positive and diminishing marginal products in all arguments, and satisfy \( F_i(0, 0) = 0 \) and \( G_i(0) = 0 \).

The profits of the firm \( k \) in sector \( i \) are given by

\[
\pi_{i,k} = (1 - \tau_i) p_{i,k} y_{i,k} - W \ell_{i,k} - \sum_{j \in I} p_j x_{ij,k},
\]

where \( p_{i,k} \) is the price at which it sells its own output, \( W \) is the wage, \( p_j \) is the price at which it purchases input \( x_{ij,k} \), and \( \tau_i \) is a sector-specific revenue tax (or subsidy).

Firms within a sector are monopolistically competitive. To facilitate this, within each sector \( i \in I \) we assume there is a producer who aggregates sectoral goods according to the following
CES production function with elasticity of substitution \( \theta_i > 0 \),

\[
y_i = \left[ \int y_{i,k}^{\frac{\theta_i - 1}{\theta_i}} dk \right]^{\frac{\theta_i}{\theta_i - 1}}.
\] (3)

The aggregator firm is perfectly-competitive, i.e. it takes all prices as given, and maximizes profits given by \( \pi_i = p_i y_i - \int p_{i,k} y_{i,k} dk \) where \( p_i \) is the price of good \( i \). We include this producer—which has zero value added and makes zero profits—for exposition only: it ensures that a homogenous good is produced by each industry, while at the same time allowing for monopolistic competition among firms within each industry.

**The household.** The preferences of the representative household are given by

\[
U(C) - V(L),
\] (4)

where \( C \) is final consumption and \( L \) is labor supply. We assume the typical regularity conditions on \( U : \mathbb{R}_+ \to \mathbb{R}_+ \) and \( V : \mathbb{R}_+ \to \mathbb{R}_+ \): they are twice continuously-differentiable with \( U', V' > 0, U'' < 0, V'' > 0 \), and satisfy the Inada conditions.

The final consumption good is a composite of sectoral goods given by:

\[
C = C(c_1, \ldots, c_N),
\] (5)

where \( c_i \) is the household’s consumption of sectoral good \( i \). We again impose that the consumption aggregator function \( C : \mathbb{R}_+^N \to \mathbb{R}_+ \) is Neoclassical: it is homogenous of degree 1, is twice continuously-differentiable in all arguments, has positive and diminishing marginal products in all arguments, and satisfies \( C(0) = 0 \).

The household owns all firms; its budget constraint is given by

\[
\sum_{i \in I} p_i c_i \leq WL + \sum_{i \in I} \left[ \int \pi_{ik} dk + \pi_i \right] + T.
\]

Total expenditure on consumption must be less than income; the latter includes wage income, dividends from firms, and lump-sum transfers, \( T \). We let the final good, \( C \), be the numeraire in this economy and hence normalize its price to one.

**Tax rebates and market clearing.** There are two types of primitive distortions in the economy: revenue taxes and monopolistic markups. We use these as simple accounting devices for any type of sectoral distortion. It will however matter for allocations whether the payments or “rents” resulting from these distortions, i.e. tax revenue or profits, are rebated back to the household or instead “thrown out.”
To allow for the latter possibility we assume that only a fraction $1 - \delta_i$ of tax revenue collected from sector $i$ is rebated lump-sum to the household so that $T = \sum_{i \in I} (1 - \delta_i) \tau_i y_i$. The remainder $\delta_i \in [0, 1)$ is thrown out; we let $h_i \equiv \delta_i \tau_i y_i$ denote the output that is discarded.

As a result, the uniform commodity produced by sector $i$ may be either consumed by the household, used as intermediate goods, or thrown out. Goods market clearing is thus given by

$$y_i = c_i + h_i + \sum_{j \in I} x_{ji}, \quad \forall i \in I,$$

where $x_{ji} \equiv \int x_{ji,k}dk$ is the total amount of good $i$ purchased by all firms in sector $j$. Labor-market clearing satisfies $L = \sum_{i \in I} \ell_i$, with $\ell_i \equiv \int \ell_{i,k}dk$.

**Remarks.** The model presented in this section is a relatively standard, multi-sector, general equilibrium, static model of intermediate good trade with generic constant returns-to-scale technologies. Taxes and markups are moreover standard accounting devices for distortions; generally it is assumed that all payments or “rents” associated with such distortions—tax revenue or profits—are rebated lump-sum to the household; see e.g. Restuccia and Rogerson (2008); Hsieh and Klenow (2009); Jones (2011, 2013); Baqaee and Farhi (2020).

The only “non-standard” feature of the model is that we permit some payments from distortions to be thrown out rather than rebated. Once one opens the door to this possibility, the question arises of which “real good” is disposed of in this circumstance. Here we have made the modeling choice that it is the particular intermediate good being taxed that is removed from the economy; this is reflected in the resource constraints in (6). A different approach is taken by Liu (2019). In his model, all payments from distortions are discarded; however, they exit the economy in the form of the final consumption good.

Finally, although the model is static, we will later on reinterpret it in a way that sheds light on the effects of within-period financing frictions.

### 3 Equilibrium Definition and Characterization

We will consider the equilibrium effects of shocks to sectoral productivities and distortions. We begin by first defining the aggregate state. Let $z \equiv (z_1, \ldots, z_N)'$ denote the vector sectoral productivities, $\tau \equiv (\tau_1, \ldots, \tau_N)'$ and $\theta \equiv (\theta_1, \ldots, \theta_N)'$ the vectors of taxes and elasticities of substitution, respectively, and $\delta \equiv (\delta_1, \ldots, \delta_N)'$ the vector of waste parameters. We assume all of these objects are exogenous and stochastic.

We let $s \equiv (z, \tau, \theta, \delta)$ denote the aggregate state; we let $S$ denote all possible realizations of the state. We represent an allocation in this economy, $\xi \equiv (\xi(s))_{s \in S}$, as a set of functions,

$$\xi(s) \equiv \left\{ (y_{i,k}(s), \ell_{i,k}(s), x_{i,k}(s))_{k \in [0,1]}, y_i(s), c_i(s), h_i(s))_{i \in I}, C(s), L(s) \right\},$$
that map the realization of the state to intermediate input and labor quantities, quantities produced, quantities consumed, quantities thrown out, and aggregate consumption and labor. We similarly represent a price system \( \varrho \equiv (\varrho(s))_{s \in S} \) in this economy as a set of functions,

\[
\varrho(s) \equiv \left\{ (p_{ik}(s))_{k \in [0,1], i \in I}, W(s), 1 \right\},
\]

that map the realization of the state to firm prices, sectoral prices, the wage, and the price of the final good normalized to one.

An equilibrium in this economy is defined as follows.

**Definition 1.** An equilibrium is an allocation and a price system, \((\xi, \varrho)\), such that for any realization of the state \(s \in S\): (i) firms’ output and input choices maximize firm profits; (ii) given prices, the household’s consumption bundle and labor supply are chosen optimally, and the household’s budget constraint is satisfied; and (iii) markets clear.

The set of equilibrium allocations may be characterized as follows.

**Proposition 1.** An allocation \(\xi\) is part of an equilibrium if and only if there exists a set of functions \(\phi_i, \psi_i, q_i : S \to \mathbb{R}_+\), for all \(i \in I\) defined by:

\[
\phi_i(s) \equiv \frac{\theta_i(s) - 1}{\theta_i(s)} \left( 1 - \frac{\tau_i(s)}{1 - \delta_i(s) \tau_i(s)} \right), \quad \psi_i(s) \equiv 1 - \delta_i(s) \tau_i(s), \quad \text{and} \quad q_i(s) \equiv \psi_i(s) y_i(s) = \psi_i(s) z_i(s) F_i(\ell_i(s), x_i(s)),
\]

such that the allocation and these functions jointly satisfy the following set of conditions:

\[
V' (L(s)) = \phi_i(s) U'(C(s)) \frac{\partial C(s)}{\partial c_i(s)} \frac{\partial q_i(s)}{\partial \ell_i(s)}, \quad \forall i \in I, k \in [0,1], s \in S
\]

\[
\frac{\partial C(s)}{\partial c_j(s)} \frac{\partial C(s)}{\partial c_i(s)} = \phi_i(s) \frac{\partial q_i(s)}{\partial x_i(s)} \frac{\partial G_i(s)}{\partial x_{ij}(s)}, \quad \forall i, j \in I, k \in [0,1], s \in S,
\]

and resource constraints

\[
L(s) = \sum_{i \in I} \ell_i(s), \quad \text{and} \quad q_i(s) = c_i(s) + \sum_{j \in I} x_{ji}(s), \quad \forall i \in I.
\]

**Proof.** See the appendix, 8.1.

Proposition 1 fully characterizes the set of equilibrium allocations by introducing two functions, \(\phi_i(s)\) and \(\psi_i(s)\), as reduced-form transformations of the underlying distortions.

The function \(\psi_i(s)\) defined in (7) represents the fraction of resources thrown out. If we redefine quantities of production from \(y_i(s)\) to \(q_i(s)\) via equation (8), then \(\psi_i(s)\) is isomorphic to a productivity shock, \(z_i(s)\). This is because the private incentives of a firm are affected in
the same way by a distortion as a productivity shock. Moreover, from a social standpoint these resources are lost. This consideration allows one to interpret \( q_i(s) \) as produced quantities net of wasted resources, while \( y_i(s) \) represents gross production. Resource constraints may thereby be restated as in (11): net production must equate total usage of the good as consumption and intermediates. This transformation thereby makes evident the allocational equivalence between \( \psi_i(s) \) and \( z_i(s) \).

Next consider the set of equilibrium conditions in (9) and (10). As all firms within a sector are identical, they make identical choices; we henceforth drop the subscript \( k \). The first set of equilibrium conditions in (9) indicate that the marginal rate of substitution between consumption of any good and labor is equated to their marginal rate of transformation modulo a wedge denoted by \( \phi_i(s) \). The second set of equilibrium conditions in (10) similarly indicates that the marginal rate of substitution between any two goods is equated to their marginal rate of transformation, modulo the same wedge.\(^4\)

In both sets of equations, the wedge \( \phi_i(s) \) is the result of the revenue tax and the monopolistic markup. Note that it is defined in (7) in terms of net quantities rather than gross.\(^5\)

In summary, a simple change-of-variables makes evident the allocational distinction between distortions whose proceeds are rebated to the household, \( \phi_i \), and those whose proceeds are not, \( \psi_i \). The former, which we herein refer to as the reduced-form “distortions” in our analysis, will be our main focus. The latter, which we herein refer to as the reduced-form “wasted distortions,” are isomorphic to sectoral productivity shocks in our setting: in either case more inputs are needed for the same level of net output.

**Financial frictions as distortions.** Distortions have thus far been microfounded as markups and taxes. In our quantitative application we will consider distortions of a different source: within-period financing frictions. To anticipate this reinterpretation, here we present the following extension of our baseline economy with working capital constraints; we call this extension the “financial frictions economy.”

To simplify, we abstract from revenue taxes: \( \tau_i(s) = 0 \) for all \( i \in I, s \in S \). Firms face a limited enforcement constraint in which they must pay for all working capital costs, i.e. intermediate good purchases and the wage bill, upfront at the beginning of the period. As firm sales are not

\(^4\) The left hand side of equation (9) is the marginal cost of labor, i.e. the household’s marginal disutility. The right hand side is the marginal benefit of labor times the distortionary wedge. The marginal benefit consists of three multiplicative components: the marginal product of labor in the production of commodity \( i \), the marginal product of good \( i \) in the production of the final good, and the household’s marginal utility of consumption of the final good. Similarly, the left hand side of equation (10) is the marginal rate of substitution of good \( i \) for good \( j \). The right hand side is the marginal product of good \( i \) in the production of good \( j \) times the distortionary wedge.

\(^5\) The primitive wedge in terms of gross quantities is \( \frac{\delta_i(s)}{\delta_i(s)(1 - \tau_i(s))} \). However, due to the fact that some tax proceeds are thrown out, \( \phi_i(s) \) defined in (7) recasts the wedge in terms of the net quantities.
realized until the end of the period, firms must take out an intra-period loan in order to finance its working capital. Specifically, firms borrow from a banking sector at an exogenous sector-specific within-period interest rate of \(1 + r_i\), implying end-of-period profits \(\pi_{i,k} = p_{i,k}y_{i,k} - (1 + r_i)(W\ell_i + \sum_{j \in I} p_jx_{ij,k})\). We furthermore assume that all interest payments accrued to the banking sector are rebated lump-sum to the household in the form of bank profits. Equilibrium allocations in this economy are characterized as follows.

**Proposition 2.** In the financial frictions economy, an allocation \(\xi\) is part of an equilibrium if and only if the conditions in Proposition 1 hold, with \(\phi\) and \(\psi\) redefined as follows:

\[
\phi_i(s) \equiv \left(\frac{\theta_i(s) - 1}{\theta_i(s)}\right) \frac{1}{1 + r_i(s)} \quad \text{and} \quad \psi_i(s) \equiv 1, \quad \forall i \in I, s \in S.
\]

**Proof.** See the appendix, 8.2.

Therefore, distortions may likewise emerge from working capital constraints. In this interpretation, financial friction severity is related to the within-period cost of borrowing: if a sector’s cost of borrowing is high, then production choices of firms within that sector are more distorted. Furthermore, if a portion of bank profits were to be thrown out rather than paid to the household, then this portion would be realized in the wasted distortion.

**Reducing the aggregate state.** Given the allocational equivalence between wasted distortions and productivity shocks, we redefine net productivity in sector \(i\) as

\[
a_i(s) \equiv \psi_i(s) z_i(s), \quad \forall i \in I, s \in S.
\]

so that net quantities satisfy: \(q_i(s) = a_i(s)F_i(\ell_i(s), x_i(s))\). Herein, all analysis of the equilibrium effects of productivity shocks will be made with respect to movements in \(a_i\), with the understanding that these results apply equally well to both the “true” underlying productivity shocks \(z_i\), as well as the wasted distortions \(\psi_i\).

In light of this transformation, we henceforth let \(a = (a_1, \ldots, a_N)'\) and \(\phi = (\phi_1, \ldots, \phi_N)'\) denote the reduced-form vectors of sectoral productivities and distortions, respectively, and we reduce the aggregate state to \(s = (a, \phi)\). In what follows we ask how these two types of sectoral shocks—productivities and distortions—manifest at the aggregate level.

### 4 Aggregation

In this section we derive a mapping from the multi-sector, input-output economy described above to the prototypical representative firm, representative household economy. We use this representation to characterize how sectoral distortions manifest at the aggregate level.
The prototype economy. We define a static, prototype economy following Chari, Kehoe, and McGrattan (2007). The prototype economy is a representative firm, representative household economy with a constant returns-to-scale (CRS) aggregate production function.

Definition 2. Let $C(s)$ and $L(s)$ be the equilibrium quantities of aggregate consumption and aggregate labor in state $s$. We define two functions $A : S \rightarrow \mathbb{R}_+$ and $\Lambda : S \rightarrow \mathbb{R}_+$ such that these equilibrium aggregate quantities satisfy:

\begin{align}
C(s) &= A(s)L(s), \quad s \in S, \\
\frac{V'(L(s))}{U'(C(s))} &= \Lambda(s) \frac{dC(s)}{dL(s)}, \quad \forall s \in S,
\end{align}

in the prototype economy. We call $A(s)$ the efficiency wedge, or total factor productivity (TFP), in state $s \in S$. We call $\Lambda(s)$ the labor wedge in state $s \in S$.

For any state $s \in S$, equations (13) and (14) are two equations in two unknowns; hence the functions $A(s)$ and $\Lambda(s)$, are well-defined. We define these two functions in such a way so that they represent aggregate wedges in the equilibrium of the prototype economy.

The equilibrium of the static prototype economy is summarized by a production function (13) and an intratemporal condition (14). The aggregate production function describes how aggregate labor is transformed into aggregate consumption; note that with CRS technologies at the micro level, the aggregate production function will indeed be CRS. The efficiency wedge $A(s)$, also known as total factor productivity (TFP), captures multiplicative shifts in the production function that resemble aggregate movements in technology.

The intratemporal condition (14) relates the economy's marginal rate of substitution between consumption and labor to the economy's marginal rate of transformation. The labor wedge, $\Lambda(s)$, acts like a tax in this dimension: it distorts the marginal rate of substitution away from the marginal rate of transformation. It thereby resembles a tax on labor income, a tax on consumption, or an aggregate markup, in the benchmark prototype economy.

The prototype economy is a “macro” representation of the economy; movement in its wedges are a way of “accounting” for business cycle fluctuations (Chari, Kehoe, and McGrattan, 2007). An extensive literature documents significant movement of the labor wedge—equivalently, the aggregate markup—at business cycle frequency (see e.g. Hall, 1997; Rotemberg and Woodford, 1999; Shimer, 2009; Karabarbounis, 2014), and moreover finds that the efficiency and labor wedge together account for the bulk of business cycle fluctuations in the postwar period (Chari, Kehoe, and McGrattan, 2007). The goal of the remainder of this section is to characterize how these aggregate wedges depend, to a first-order, on the underlying sectoral shocks.
4.1 The macroeconomic effects of sectoral shocks

Without parametrically specifying the economy's preferences, technologies, and input-output structure, it is impossible to obtain analytical, closed-form solutions for the aggregate wedges described in the previous subsection as functions of the underlying shocks. However, quite a lot can be said about the macroeconomic effects of sectoral shocks to a first-order without requiring these specifications; we turn to this next.

Definition 3. We define the efficient steady-state in this economy as the state in which $a = 1 \equiv (1, 1, \ldots, 1)'$ and $\phi = 1 \equiv (1, 1, \ldots, 1)'$. For all sectors $i \in I$, we define

$$\lambda_i \equiv \frac{p_i y_i}{C},$$

to be the steady-state equilibrium Domar weight of industry $i$.

We begin by defining the efficient steady-state to be the state in which all sectoral productivities and distortions are set equal to 1. For the remainder of this section, we will derive results regarding the macroeconomic effects of shocks in the neighborhood of this steady state. Note that setting all productivities equal to 1 is simply a normalization. However, setting all distortions equal to 1 has economic meaning: we assume efficiency at the steady-state equilibrium; this effectively rules out any steady-state taxes and/or markups.

Definition 3 furthermore defines the Domar weights as sectoral sales as a share of GDP. In any economy with intermediate goods, the sum of these weights across sectors is greater than 1 as gross sales exceeds total value added. This definition facilitates our statement of the following well-known theorem known as Hulten's theorem.

Theorem 1. (Hulten, 1978) In an efficient economy, sectoral productivities have first-order effects on TFP equal to their Domar weights:

$$\frac{\partial \log A}{\partial \log a_i} = \lambda_i, \quad \forall i \in I.$$

Proof. See the appendix, 8.3. \qed

Hulten's theorem states that in an efficient economy, the Domar weights are sufficient statistics for the first-order effects of sectoral productivities on TFP (Hulten, 1978). Note that in our context it applies not only to the aggregate effects of the true underlying productivity shocks $z_i$, but also with respect to the wasted distortions $\psi_i$.

This theorem tells us that to a first-order, knowledge of the microeconomic intricacies and “details” governing the structure of an economy—its input-output network, the parametric structure of its technologies, and the household’s preferences over goods—is unnecessary for inferring the first-order response of TFP to sectoral productivities. Sectoral sales, and hence Domar weights, are easily observable and suffice.
We now turn to the macroeconomic effects of sectoral distortions. Unlike productivity shocks, distortions do not actively create or destroy resources; they simply drive wedges between prices and marginal costs. By moving production decisions away from their optima, distortions result in an inefficient reallocation of resources across the input-output network, and hence a loss in aggregate productive efficiency. Furthermore, the aggregate effects of individual wedges compound as firms buy and sell from one another within the network of intermediate good trade. As a result, the price of the final consumption good is distorted away from its marginal cost of production, and a labor wedge arises.

Again, without parametrically specifying the economy’s primitives, it is impossible to obtain analytical, closed-form solutions for the labor and efficiency wedges as functions of the underlying sectoral distortions. Instead, the following result is derived from a first-order approximation of these aggregate objects around the efficient steady-state.

**Theorem 2.** Starting from the efficient steady state, sectoral distortions have: (i) zero first-order effects on TFP:

$$\frac{d \log A}{d \log \phi_i} = 0, \quad \forall i \in I,$$

and (ii) non-zero first-order effects on the labor wedge equal to sectoral Domar weights:

$$\frac{d \log \Lambda}{d \log \phi_i} = \lambda_i, \quad \forall i \in I.$$

**Proof.** See the appendix, 8.4.

Part (i) of Theorem 2 should come as no surprise. The proof is straightforward: for any vector of productivities, the equilibrium allocation at $\phi = 1$ maximizes productive efficiency in the Diamond and Mirrlees (1971) sense. Therefore, starting from this maximum, small distortions have no first-order effects in productive efficiency. Misallocation manifests in second and higher-order terms alone; see also Jones (2013); Baqaee and Farhi (2020).

But consider now part (ii) of Theorem 2. This part resembles Hulten’s Theorem but for the macroeconomic effects of sectoral distortions. Part (ii) states that to a first-order near efficiency, knowledge of the underlying microeconomic details of the economy—technologies, preferences, and input-output structure—are again unnecessary insofar as sales shares are observable. Domar weights serve as sufficient statistics for the first-order labor wedge response to sectoral distortions.

In total, Theorem 2 characterizes the first-order macroeconomic response to sectoral distortions. Movements in underlying sectoral distortions manifest at the aggregate level via two channels: the efficiency wedge and the labor wedge. Our results suggest that in economies with little to no misallocation, the labor wedge will likely be the quantitatively stronger channel by which these distortions manifest, relative to TFP. We return to this implication in Section 6.
Remarks and Implications. Theorem 2 contributes to our understanding of the mapping from micro shocks to macroeconomic fluctuations.

Following the seminal work of Gabaix (2011), a large literature studies the aggregate effects of “granular” productivity shocks (see e.g. Foerster et al., 2011; Acemoglu et al., 2012; Carvalho and Gabaix, 2013; di Giovanni et al., 2014, 2018, among others). One implication of Theorem 2 is that the results of this work can in theory be extended to analyzing the first-order labor wedge effects of “granular” distortions.

Specifically, Gabaix (2011) shows that in an efficient economy, idiosyncratic firm-level productivity shocks can result in macroeconomic fluctuations when the firm size distribution is sufficiently fat-tailed; in this case, idiosyncratic productivity shocks do not wash out. In an extension, Gabaix (2011) further demonstrates that this result continues to hold in an input-output economy as long as firm sales are sufficient statistics for the first-order aggregate effects of productivity shocks, thereby invoking Hulten (1978). Theorem 2 resembles Hulten’s theorem, suggesting that the Gabaix (2011) result can be extended in the following manner: idiosyncratic shocks to firm-level markups can result in first-order labor wedge movement as long as the firm size distribution is fat-tailed.

Theorems 1 and 2 also provide guidance on the reverse mapping: how movements in macro aggregates relate to underlying micro shocks. Consider a macro-econometrician studying aggregate fluctuations and suppose this econometrician were to observe a recession in which TFP falls significantly while the labor wedge remains constant. In the absence of any obvious “macro shock,” our results suggest that the most likely “micro” explanation is negative productivity shocks hitting large sectors or firms, as opposed to movements in distortions. On the other hand, if the econometrician were to observe a precipitous dip in the labor wedge but no movement in TFP, our results point to a different micro source. In this case, shocks to distortions in large sectors or firms, as opposed to productivities, are the likely cause.

As stressed in Chari, Kehoe, and McGrattan (2007), business cycle accounting via aggregate wedges does not necessarily inform one on the micro origins of these movements. Our results make progress in this dimension by providing, in a relatively general (albeit static) input-output economy, a first-order mapping between the economy’s macro fluctuations and their likely “micro culprits.” In particular, sectoral productivities lead to first-order movements in TFP, while sectoral distortions lead to first-order movements in the labor wedge. While this heuristic may sound ex-post obvious, our work provides a formal, theoretical foundation for this first-order “micro to macro” mapping.

We close this discussion with three remarks. First, the results presented in Theorem 2 are stated as approximations around an efficient steady-state; we do this in order to mirror Hulten’s theorem and to define Domar weights at their steady-state values for quantitative purposes. However, what is necessary for our result is that we approximate in the neighborhood of an
efficient economy, but this economy need not be in steady-state. We can generalize Theorem 2 so that the underlying efficient economy is stochastic, in which case the appropriate Domar weights would be the stochastic sales shares in the efficient economy.\(^6\)

Second, let us stress that Theorem 2 holds only up to a first-order around efficiency, i.e. for small shocks. To the extent that the economy is away from efficiency, these approximations are no longer valid. In this case, distortions begin to have first-order effects on TFP; see Baqaee and Farhi (2020).\(^7\)

Finally, Domar weights are equilibrium objects—they do not provide a structural, economic interpretation for how they are determined in equilibrium by the economy’s primitives: technologies, preferences, and in particular, the input-output structure. In what follows, we proceed to restrict attention to economies with Cobb-Douglas preferences and technologies. Within this class, we provide such an economic interpretation.

### 4.2 The Cobb-Douglas Economy special case

We now consider the special case of the economy with Cobb-Douglas technologies and preferences; we specify these as follows.

\[
F_i(\ell_i, x_i) = \ell_i^{\alpha_i}(x_i)^{1-\alpha_i}, \quad G_i(x_i) = \prod_{j \in I} x_{ij}^{g_{ij}}, \quad \text{and} \quad C(c_1, \ldots, c_N) = \prod_{i \in I} c_i^{v_i}. \quad (15)
\]

where \(\alpha_i\) is the labor share of sector \(i\), \(g_{ij} \in [0, 1]\) denotes the share of good \(j\) in the intermediate good composite of sector \(i\), and \(v_i \in [0, 1]\) is the household’s expenditure share on good \(i\). To maintain homogeneity of degree 1, we impose \(\sum_{j \in I} g_{ij} = 1\), for all \(i \in I\), and \(\sum_{j=1}^{N} v_j = 1\). This economy is similar to the one studied in Jones (2013), but with elastic labor supply.

In vector notation, we let \(\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N)'\) denote the vector of sectoral labor shares and \(v = (v_1, v_2, \ldots, v_N)'\) denote the vector of household expenditure shares. We let \(G\) denote the \(N \times N\) input-output matrix with entries \(g_{ij}\); the rows of \(G\) sum to one.\(^8\) Finally, for any vector \(w \in \mathbb{R}_+^N\), we define \(\text{diag}(w)\) to be the square matrix with the elements of vector \(w\) on the main diagonal and all off-diagonal elements equal to zero.

We begin our analysis of the Cobb-Douglas economy with an intermediate lemma. Let \(p(s) \equiv (p_1(s), \ldots, p_N(s))'\) denote the equilibrium price vector.

**Lemma 1.** In the Cobb-Douglas economy, equilibrium prices and the equilibrium wage satisfy:

\[
\log p(s) - \log W(s) = -L [\log a(s) + \log \phi(s)] + \kappa^p, \quad \forall s \in S, \quad (16)
\]

\(^6\)To see this, note that in our proof in 8.4 we approximate around an arbitrary state \(s_0\). We then use only the efficiency properties of the state to reduce terms and obtain the desired result.

\(^7\)Baqaee and Farhi (2020) provide an elegant characterization of the first-order effects of distortions and productivity shocks on TFP both at and away from efficiency. With respect to their work, we instead characterize the aggregate effects of distortions on the labor wedge near efficiency.

\(^8\)The column sum of \(G\) is its weighted outdegree: the share of sector \(i\)’s output in the input uses of the other sectors in the economy.
where \( \mathbf{L} \) is the Leontief inverse matrix, defined by
\[
\mathbf{L} \equiv \left[ \mathbb{I}_N - \text{diag} \left( \mathbf{1} - \alpha \right) \mathbf{G} \right]^{-1},
\]
and \( \kappa^p \) is a vector of constants.

**Proof.** See the online appendix, Section 1.3.

Lemma 1 characterizes the equilibrium vector of sectoral prices relative to the wage in this economy. This result indicates that the two types of sectoral shocks—productivities and distortions—are isomorphic in terms of their relative price effects. Furthermore, equilibrium prices reflect not only the direct price effect of a shock to its original sector, but also all indirect price effects resulting from the propagation of the shock to the input costs of the sector’s customers, customers’ customers, and so on.

To see this, one may expand the Leontief inverse matrix in expression (16) as follows:
\[
\left[ \mathbb{I}_N - \text{diag} \left( \mathbf{1} - \alpha \right) \mathbf{G} \right]^{-1} = \mathbb{I}_N + \text{diag} \left( \mathbf{1} - \alpha \right) \mathbf{G} + \left( \text{diag} \left( \mathbf{1} - \alpha \right) \mathbf{G} \right)^2 + \cdots .
\]

Consider a negative shock to sector \( i \). The first term in this expansion, \( \mathbb{I}_N \), yields the direct effect of the shock to this sector: sectoral output falls and in equilibrium its price rises. This direct price effect is the same regardless of whether the initial shock to sector \( i \) is a productivity shock or a distortion.

But now note that this sectoral price increase raises the input price for all downstream customers of sector \( i \)—that is, the sectors that use sector \( i \)’s intermediate good. In response, these sectors purchase less of that input, produce less output and, in equilibrium, raise prices. This secondary, indirect price effect of sector \( i \)’s customers is captured in the second term in the above expansion, \( \text{diag} \left( \mathbf{1} - \alpha \right) \mathbf{G} \). But now note that their price changes raise the input prices of their own downstream customers. In response, these sectors also purchase less inputs, produce less output, and, in equilibrium, raise prices. This third, indirect price effect of sector \( i \)’s customers’ customers is captured in the third term in the above expansion, and so on.

To summarize, the Leontief Inverse matrix captures the entire infinite hierarchy of all higher-order network effects for productivity shocks and distortions. The extent to which these shocks propagate through the network depends crucially on the input-output network structure embedded in \( \mathbf{G} \). Moreover, equilibrium price effects are the same regardless of whether the original shock is a distortion or a productivity.

But how does this network propagation translate to allocations and, in particular, aggregate fluctuations? Unlike productivity shocks, distortions do not create or destroy resources: they merely move around allocations while all production possibilities remain intact. Thus, while distortions and productivities may be isomorphic in terms of equilibrium prices effects, they are not isomorphic in terms of equilibrium allocations. This is evidenced in the following result.

**Theorem 3.** The Cobb-Douglas economy aggregates to the prototype representative household, representative firm economy as defined in Definition 2.
(i) Globally, sectoral productivity shocks have zero effects on the labor wedge and the following non-zero effects on TFP:

\[ \frac{d \log A}{d \log a_i} = \beta_i \quad \forall i \in I, \tag{17} \]

where \( \beta_i \) is the \( i \)th element of the following vector:

\[ \beta' \equiv \mathbf{v}' \mathbf{L} = \mathbf{v}' \left[ \mathbb{I}_N - \text{diag} (1 - \alpha) \mathbf{G} \right]^{-1}. \tag{18} \]

(ii) Starting from the efficient steady state, sectoral distortions have zero first-order effects on TFP and the following non-zero first-order effects on the labor wedge:

\[ \frac{d \log \Lambda}{d \log \phi_i} = \beta_i, \quad \forall i \in I. \tag{19} \]

Proof. See the online appendix, Section 1.2.

Theorem 3 is akin to Theorems 1 and 2 in that it maps sectoral shocks to first-order movements in aggregate wedges, but for the special case of the Cobb-Douglas economy.

In this economy, Theorems 1 and 2 still hold: Domar weights remain sufficient statistics for the first-order effects of productivities on the efficiency wedge as well as the first-order effects of distortions on the labor wedge. However, Domar weights are equilibrium objects. Theorem 3 provides a structural, economic interpretation for how they are determined in equilibrium: in the Cobb-Douglas economy, these objects can be expressed in closed-form as functions of the economy’s underlying technologies, preferences, and input-output structure.

Consider first part (i) of Theorem 3; this result may originally be attributed to Burress (1994), but can also be found more recently in Acemoglu et al. (2012). Equation (17) provides a succinct, closed-form expression for the elasticity of TFP to a sectoral productivity shock. In the Cobb-Douglas economy, this elasticity is equal to \( \beta_i \), the \( i \)th element of what Acemoglu et al. (2012) term the “influence vector,” \( \beta \). This vector, defined in (18), is a simple transformation of the Leontief inverse matrix into aggregate consumption.\(^9\)

As described above, the Leontief inverse matrix captures the infinite hierarchy of all higher-order network effects. It is closely related to the mathematical notion of Bonacich centrality: a measure of “importance” within a network. Therefore, in the Cobb-Douglas economy, productivity shocks in sectors with high centrality—sectors with more direct and/or indirect downstream customers—have greater effects on TFP than productivity shocks in sectors with low centrality.\(^10\)

Note that unlike Hulten’s theorem, equation (17) is not an approximation—it is exact and moreover holds globally, not just in the neighborhood of the efficient steady state; this property

\(^9\)Formally, \( \beta_i \equiv e_i' \beta \) where \( e_i \) denotes a column vector of length \( N \) with the \( i \)th element equal to one and zero otherwise.

\(^10\)See Acemoglu et al. (2012); Carvalho (2014); Carvalho and Tahbaz-Salehi (2019) for more details.
is due to the Cobb-Douglas specification. Moreover, in our context it applies not only to the aggregate effects of the true underlying productivity shocks, $z_i$, but also with respect to the wasted distortions, $\psi_i$.

Part (ii) of Theorem 3 likewise provides an explicit, closed-form expression for the first-order effects of distortions on the labor wedge in terms of the economy's underlying technologies, preferences, and input-output structure. Unlike part (i), this result holds only up to first-order around efficiency; it does not hold globally. In particular, part (ii) states that near efficiency, the first-order effect of a sectoral distortion on the labor wedge is likewise equal to $\beta_i$, the $i$th element of the “influence vector.”

This result implies that the same economic intuition for how sectoral productivity shocks affect TFP applies equally well to how small distortions affect the labor wedge. Specifically, in the Cobb-Douglas economy, distortions in sectors with high centrality—sectors with more direct and/or indirect downstream customers—have greater effects on the labor wedge than distortions in sectors with low centrality.

Recall that the network propagation of both sectoral distortions and productivity shocks work through movements in prices. Theorem 3 thereby translates this price movement into aggregate allocations: sectoral productivities result in aggregate fluctuations in the efficiency wedge, while sectoral distortions manifest in first-order movement in the labor wedge. Finally, by combining the results of Theorem 3 with those of Theorems 1 and 2, we can infer that in the special case of the Cobb-Douglas economy, the efficient steady-state Domar weights, $\lambda_i$, are equal to $\beta_i$, for all $i \in I$.

5 Supply Chains and Bottleneck Sectors

While the results for small distortions may be helpful for intuition, they hold only up to a first-order around efficiency. To the extent that the economy is away from efficiency, Domar weights and network “centralities” are no more sufficient to capture the aggregate effects of distortions. In this section we consider a special case, a “supply chain” economy, in order to gain a better understanding for the aggregate effects of distortions away from efficiency.

Throughout this section we abstract entirely from productivity shocks and focus exclusively on the aggregate effects of distortions. Formally we set $a = 1 \equiv (1, 1, \ldots, 1)'$ for all $s \in S$, but allow the vector of distortions, $\phi$, to vary freely. For shorthand we introduce

$$
\chi_i(s) \equiv \frac{d \log \Lambda}{d \log \phi_i}(s)
$$

(20)

to denote the elasticity of the labor wedge with respect to the distortion in sector $i$ in state $s$. In the previous section we characterized these elasticities only up to a first-order near efficiency; in this section we relax both of these restrictions and characterize the total effect (including the first-order but also all higher-orders) for any state $s \in S$. 

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5.1 The Supply Chain Economy

We begin our analysis with a special case of the Cobb-Douglas economy; we call this the “supply chain economy.” Consider an economy of $N$ sectors with Cobb-Douglas production functions,

$$y_i = \ell_1^{\alpha_i} x_{i,i-1}^{1-\alpha_i}, \quad \text{where} \quad \alpha_i = \alpha^{i-1}, \quad \forall i \in I,$$

for some scalar $\alpha \in (0, 1)$. That is, every sector $i$ combines the primary factor, labor, with the intermediate good produced by its neighboring upstream sector $i-1$, to produce its output. The most upstream sector, Sector 1, uses only labor. This sector has the greatest sectoral labor share equal to one, and the sectoral labor share decreases exponentially at a rate $\alpha$ as one moves down the supply chain.

The output of each sector may be used as either an intermediate good—sold to its neighboring downstream sector—or as a final good sold to the household. The most downstream sector, Sector $N$, sells its product only to the household. Market clearing conditions are thus given by $y_i = c_i + x_{i,i+1}$, for all $i \in \{1, \ldots, N-1\}$, and $y_N = c_N$.

Finally, the household’s Cobb-Douglas consumption basket is given by:

$$C(c_1, \ldots, c_N) = \prod_{i \in I} c_i^{v_i}, \quad \text{with} \quad v_i = \rho^{N-i}/\varrho, \quad \forall i \in I,$$

for some scalar $\rho \in (0, 1)$; we set the denominator $\varrho = \sum_{j=1}^{N} \rho^{N-j}$ in order to normalize the sum of the consumption expenditures shares to one. The the most downstream sector, Sector $N$, has the greatest consumption share equal to $1/\varrho$, and the consumption share of each sector decreases exponentially at a rate $\rho$ as one moves up the supply chain.

The supply chain economy is thereby parameterized by only two scalars, $\alpha$ and $\rho$. For any arbitrary pair $(\alpha, \rho) \in (0, 1)^2$, there are multiple paths by which the primary factor, labor, is transformed into the final consumption; see the first panel of Figure 1.

The pure vertical economy. By varying $\alpha$ and $\rho$, we can control the topology of these labor-to-consumption paths. To see this, we first consider an extreme case: a limit economy in which
all paths reduce to one.

**Proposition 3. The pure vertical economy.** Let $\alpha \to 0$ and $\rho \to 0$. In this limit, the efficiency wedge and labor wedge are given by:

$$A(s) = 1 \quad \text{and} \quad \Lambda(s) = \prod_{i \in I} \phi_{i}(s), \quad \forall s \in S.$$

The elasticities of the labor wedge with respect to sectoral distortions satisfy $\chi_{i}(s) = 1$, for all $i \in I, s \in S$.

**Proof.** See the online appendix, Section 2.2.

In this limit, the supply chain economy converges to what we call the “pure vertical economy.” Sectors are arranged in a vertical supply chain in which only the first sector, Sector 1, hires labor, while each consecutive sector uses only the intermediate good produced by its neighboring upstream sector as its input. At the end of the supply chain, Sector $N$ produces the unique final good consumed by the household.\(^{11}\)

The pure vertical economy is visually represented in the second panel of Figure 1; it is is similar to what are known as “snake” economies in the global value chains literature (Antrás et al., 2017; Antrás and de Gortari, 2017; Baldwin and Venables, 2013). Unlike the more generic supply chain economy, the pure vertical limit economy has only one route by which the primary factor, labor, is transformed into household consumption.\(^{12}\)

As a result, in this limit economy there is no room for misallocation: inputs can in no way be misallocated away from their unique path. TFP is thereby independent of all sectoral distortions and, in the absence of productivity shocks, is constant.

On the other hand, while sectoral distortions do not generate any loss in productive efficiency, they do induce a labor wedge. As labor and inputs move downstream along the pure vertical supply chain, sectoral output prices are distorted away from marginal costs. This process compounds and as a result, the price of the final consumption good is distorted away from its true marginal cost of production; by implication, a labor wedge—equal to the product of sectoral distortions—arises.

To summarize, in the pure vertical economy, the efficiency wedge is independent of sectoral distortions, while the elasticity of the labor wedge with respect to any sectoral distortion is equal to one. These properties hold not only at the efficient allocation but also globally, i.e. for any state $s \in S$.

\(^{11}\)Formally, in this limit the vectors of sectoral labor shares and sectoral consumption shares converge to $\alpha = (1, 0, \ldots, 0)'$ and $\nu = (0, \ldots, 0, 1)'$, respectively.

\(^{12}\)The opposite extreme, the pure horizontal economy, is introduced in Section 6.
Bottlenecks. By reducing both $\alpha$ and $\rho$, the multiple labor-to-consumption paths of the supply chain economy converge to a unique path in the limit. In this limit, sectoral distortions have very powerful effects on the labor wedge but zero effect on TFP.

We next show that by varying $\alpha$ and $\rho$ independently, these lessons are robust to the existence of multiple paths as long as there exist key “bottleneck” sectors through which all labor-to-consumption paths must pass.

Proposition 4. Consider the following two limits of the hybrid chain economy.

(ii) The supply-bottleneck. Let $\alpha \to 0$ for fixed $\rho$. In this limit, the efficiency wedge $A(s)$ is a function of $\{\phi_2(s), \ldots, \phi_N(s)\}$ but is independent of $\phi_1(s)$, for all $s \in S$. The elasticity of the labor wedge with respect to sectoral distortions satisfies the following strict ordering:

$$1 = \chi_1(s) > \chi_2(s) > \cdots > \chi_N(s), \quad \forall s \in S.$$

(iii) The demand-bottleneck. Let $\rho \to 0$ for fixed $\alpha$. In this limit, the efficiency wedge $A(s)$ is a function of $\{\phi_1(s), \ldots, \phi_{N-1}(s)\}$ but is independent of $\phi_N(s)$, for all $s \in S$. The elasticity of the labor wedge with respect to sectoral distortions satisfies the following strict ordering:

$$\chi_1(s) < \chi_2(s) < \cdots < \chi_N(s) = 1, \quad \forall s \in S.$$

Proof. See the online appendix, Section 2.3.

The two intermediate limit cases presented in Proposition 4 are constructed by reducing one of the two parameters to zero, while holding the other fixed.

Consider first the limit in part (i). In this limit, labor is used only by the most upstream sector: Sector 1. However, the household consumes all sectoral goods and as such there remain multiple paths by which labor is transformed into consumption. In contrast, consider the limit featured in part (ii). In this limit, the household consumes only the good produced by the most downstream sector: Sector $N$. However, all sectors use labor as an input and as such there remain multiple paths by which labor is transformed into consumption. These economies are visually represented in Figure 2.

Despite the existence of multiple labor-to-consumption paths, in each of these economies there exists a “bottleneck” sector. We call a “bottleneck” sector a sector through which all labor-to-consumption paths must pass. In the economy in part (i), even though the household consumes all goods, all labor must first pass through Sector 1. Sector 1 thereby serves as a “supply-bottleneck.” In the economy in part (ii), although labor is used in the production of every sector, all intermediate goods are ultimately funneled through Sector $N$ en route to consumption. Sector $N$ thereby serves as a “demand-bottleneck.”

\[\text{Note that as } \alpha \to 0 \text{ the vector of labor shares approaches } \alpha \to (1, 0, \cdots, 0)'. \text{ Similarly, as } \rho \to 0 \text{, the vector of expenditure shares approaches } v \to (0, \cdots, 0, 1)'\].
Supply Bottleneck: $\alpha \to 0, \rho \in (0, 1)$

Demand Bottleneck: $\rho \to 0, \alpha \in (0, 1)$

Figure 2. Supply and Demand Bottlenecks

Proposition 4 indicates that bottleneck sectors are critical: a distortion to a bottleneck sector distorts all labor-to-consumption paths. As a result, there is 1-to-1 pass-through of a bottleneck distortion to the labor wedge in any state. Moreover, the elasticity of the labor wedge with respect to a distortion monotonically decreases as one moves away from the bottleneck sector; this strict ordering is invariant to the aggregate state.

And yet, despite the fact that bottleneck distortions have the greatest effect on the labor wedge, bottleneck distortions have zero effect on TFP. In either economy, the efficiency wedge is independent of the bottleneck sector’s distortion; as in the pure vertical economy, a distortion in this sector generates no misallocation.\(^{14}\)

5.2 Generalization of Bottlenecks to the Cobb-Douglas Economy

The particular limits of the supply chain economy of the previous subsection are stark examples that isolate the aggregate effects of distortions in bottleneck sectors. While the existence of a bottleneck sector so far appears to be rather non-generic, it turns out that one can define this concept much more generally for any Cobb-Douglas economy.

Consider the generic Cobb-Douglas economy established in Subsection 4.2. In this economy we define the following objects:

\[
b^S_i(s) \equiv \alpha' \left[ \text{diag}(\phi(s))^{-1} - G' \text{diag}(1 - \alpha) \right]^{-1} e_i, \quad \forall i \in I, s \in S, \tag{21}
\]

\[
b^D_i(s) \equiv \nu' \left[ \text{diag}(\phi(s))^{-1} - G' \text{diag}(1 - \alpha) \right]^{-1} e_i, \quad \forall i \in I, s \in S. \tag{22}
\]

where $e_i$ denotes a column vector of length $N$ with the $i$th element equal to one and zero otherwise. The objects $b^S_i(s)$ and $b^D_i(s)$ are sector and state-specific measures of supply and demand bottlenecks, respectively. To see this, note that the measure $b^S_i(s)$ is larger the more

\[^{14}\text{In this subsection we have explored different limits of the supply chain economy. There is, however, an entire continuum of economies within these limits. In Section 2.4 of the online appendix we consider intermediate values of } (\alpha, \rho) \in (0, 1)^2 \text{ and examine the comparative statics of labor wedge elasticities as we move continuously within the extremes.} \]
sector \( i \) depends on labor either directly or indirectly (through sector \( i 's \) suppliers, suppliers’ suppliers, and so on) in state \( s \in S \); this is similar to our notion of a supply bottleneck. Likewise, the measure \( b^P_i(s) \) is larger the more the household depends on the output produced by sector \( i \) either directly or indirectly (through sector \( i 's \) customers, customers’ customers, and so on) in state \( s \in S \); this is similar to our notion of a demand bottleneck.

With these definitions, we obtain the following characterization.

**Proposition 5.** In the generic Cobb-Douglas economy, the elasticity of the labor wedge with respect to a sectoral distortion satisfies:

\[
\chi_i(s) \equiv \frac{d \log \Lambda}{d \log \phi_i(s)} = \frac{1}{\phi_i(s)} \Lambda(s)^{-1} b^S_i(s)b^P_i(s), \quad \forall i \in I, s \in S,
\]

and the elasticity of the efficiency wedge with respect to a sectoral distortion satisfies:

\[
\frac{d \log A}{d \log \phi_i(s)} = \beta_i - \chi_i(s), \quad \forall i \in I, s \in S.
\]

**Proof.** See the online appendix, Section 1.4.

Proposition 5 indicates that in the Cobb-Douglas economy, a sectoral distortion has a greater effect on the labor wedge the greater its measure of being either a supply bottleneck, a demand bottleneck, or both. In contrast, a sectoral distortion has a smaller effect on the efficiency wedge the greater its measure of being either a supply bottleneck, a demand bottleneck, or both. Proposition 5 thereby provides a generalization of the lessons highlighted by the limits of the supply chain economy explored above.

**Remarks.** In any input-output economy, there exist multiple paths by which primary factors are transformed into final goods. In the absence of distortions, primary factors and intermediate goods flow through these paths in a way that maximizes Diamond and Mirrlees (1971) productive efficiency. Misallocation occurs when, and only when, distortions alter the flow of inputs through the network in a way that deviates from their optima. The extent to which these flows are misallocated is what lowers productive efficiency and, thereby, TFP.

The limits of the supply chain economy presented in this section isolate a particular lesson. Namely, there may exist critical bottleneck sectors in the economy through which most or all of these labor-to-consumption paths must pass. More generally, a sector that acts as a supply bottleneck is one which relies more heavily on the primary factor, either directly or indirectly, and a sector that acts as a demand bottleneck is one in which the household consumes more of its output, either directly or indirectly. Our results in this section provide the simple insight that distortions in bottleneck sectors can have large effects on the labor wedge while, at the same time, contribute the least to movements in TFP.
6 Quantitative Illustration

In this section we study a particular quantitative application of our framework: the network amplification of within-period financing frictions during the 2008-09 Financial Crisis. We calibrate the model to the US input-output tables and use the Gilchrist and Zakrajšek (2012) excess bond premia as our proxy for financial distortions. We then examine the model's aggregate implications. The goal of this quantitative exercise is not to explain what caused the 2008-09 Financial Crisis, but instead to simply illustrate how the US input-output structure may have amplified the macroeconomic effects of financial frictions during this period.

6.1 The Data and Calibration

We calibrate the Cobb-Douglas input-output network presented in Subsection 4.2 using the Bureau of Economic Analysis's (BEA) input-output tables. These tables are produced annually beginning in 1997 at the 3-digit North American Industry Classification System (NAICS) level. For each sector $i$ in year $t$, we observe the following: (i) total sales, the empirical analogue of $p_{it}y_{it}$, (ii) expenditure on goods produced by sector $j$ in year $t$, the empirical analogue of $p_{jt}x_{ij,t}$, (iii) labor expenses, the empirical analogue of $W_{it}\ell_{it}$, and (iv) the final use of the sector's production, our empirical analogue of $p_{it}c_{it}$.

We use this data to construct annual labor expenditure shares, final use (consumption) shares, and intermediate good expenditure shares as follows:

$$\alpha_{it} = \frac{W_{it}\ell_{it}}{W_{it}\ell_{it} + \sum_{j} p_{jt}x_{ij,t}}, \quad \forall i, t, \quad v_{it} = \frac{p_{it}c_{it}}{\sum_{i} p_{it}c_{it}}, \quad \forall i, t, \quad \text{and} \quad g_{ijt} = \frac{p_{jt}x_{ij,t}}{\sum_{j} p_{jt}x_{ij,t}}, \quad \forall i, j, t,$$

in line with the Cobb-Douglas specification in (15). These values are relatively stable from year to year; for the rest of our analysis we thus drop their time subscript and use their 2007 values, the year prior to the crisis.\(^{15}\)

A measure of financial distortions: the GZ excess bond premia. In this application we return to the financial frictions economy presented in Section 3 in which we reinterpret the distortions as the result of working capital constraints. We use the sector-level excess bond premia of Gilchrist and Zakrajšek (2012) as our measure of financial frictions.

Gilchrist and Zakrajšek (2012) construct their “GZ credit spread” as follows. First, the authors take individual fixed-income securities and discount their promised cash-flows according to zero-coupon US Treasury yields. This procedure delivers an overall credit spread for each

\(^{15}\)In Section 3.1 of online appendix, we document the cross-sectional heterogeneity in these values. We report the top 20 industries in our sample with the greatest expenditure labor shares, $\alpha_{i}$, the top 20 industries with the greatest expenditure intermediate good shares, $1 - \alpha_{i}$, and the top 20 industries with the greatest shares in final uses, $v_{i}$.
security given by the yield of the corporate bond over a hypothetical Treasury security with the exact same cash flows.

The “GZ excess bond premium” for each security is then constructed as the portion of the overall credit spread that cannot be accounted for by individual predictors of default, nor by bond-specific characteristics. Specifically, the authors regress their credits spreads on a firm-specific measure of expected default and a vector of bond-specific characteristics; the residual of this regression is the GZ excess bond premium.

With regard to the firm’s probability of default, the authors construct a distance-to-default measure at each point in time based on the Merton (1974) model. The equity of the firm can be viewed as a call option on the underlying value of the firm with a strike price equal to the face value of the firm’s debt. Assuming that the firm’s value follows a geometric Brownian motion, one may then apply the Black-Scholes-Merton option-pricing formula and calculate distance-to-default with observable asset price data. The distance-to-default should thereby summarize all available information regarding the risk of default, according to the Merton model.

We use the sector-level indices of the GZ excess bond premia, generously provided to us by the authors, as our measure of within-period financial frictions. For each sector $i$ in year $t$, we denote the GZ excess bond premia by $r^{GZ}_{it}$. We interpret this premia as the intra-period borrowing rate faced by sector $i$ in period $t$. According to Proposition 2, sectoral distortions in the model are related to the within-period cost of borrowing as follows:

$$\phi_{it} = \frac{1}{1 + r^{GZ}_{it}}.$$  \hspace{1cm} \text{(23)}

where, for the purposes of this exercise, we abstract from firm markups.\footnote{We implicitly assume $\theta_i(s) \to \infty$, for all $i \in I, s \in S$.} We thereby use equation (23) to construct our baseline measure of sectoral distortions and herein refer to these as the “GZ distortions.”

The GZ measure has several virtues that make it ideal for our application. By orthogonalizing out default probabilities, these premia are meant to capture time- and sector-varying changes in the compensation demanded by the financial sector in providing credit—above and beyond expected losses. Gilchrist and Zakrajšek (2012) find that adverse shocks to highly leveraged financial intermediaries during the 2008-09 Financial Crisis led to sharp increases in this measure. This evidence is consistent with the notion that the GZ excess bond premium reflects a reduction in the risk-bearing capacity of the financial sector, and as a result, an increase in the cost of credit extended to firms. We refer the reader to their paper for more details.

**Returns to Scale (or Factor Income Shares).** Finally, our baseline model makes the simplifying assumptions of constant returns-to-scale technologies and a single primary factor
of labor. In our quantitative analysis, we relax these assumptions and allow for sector-specific fixed factors in a reduced-form way. Specifically, we modify the production function as follows:

\[ y_{it} = z_{i,t} \left( \ell_{it}^{\alpha_i} x_{it}^{1-\alpha_i} \right)^{\eta_i}. \]  

One may interpret the new sector-specific parameter \( \eta_i \in (0, 1] \) as controlling the sector’s decreasing returns-to-scale. Equivalently, in a model with constant returns-to-scale technology but with fixed factors, \( \eta_i \) can be interpreted as the share of total income accrued to labor and intermediates. In the latter interpretation, \( 1 - \eta_i \) is the share of total income accrued to all other fixed factors, e.g. physical capital or land.

We allow \( \eta_i \) to be heterogeneous across sectors but constant over time. The question then becomes how one may calibrate this parameter. In each sector we observe total expenditure on inputs and gross sales each period. We therefore denote the expenditure-to-sales ratio for sector \( i \) in time \( t \) by \( \varepsilon_{it} \) and compute it in the BEA data as follows:

\[ \varepsilon_{it} \equiv \frac{W_{it} \ell_{it} + \sum_j p_{jt} x_{ij,t}}{p_{it} y_{it}}. \]  

In a frictionless economy with Cobb-Douglas technologies, this ratio is constant over time. With distortions, this ratio satisfies \( \varepsilon_{it} = \phi_{it} \eta_i \) and is thereby time-varying.

Unfortunately, we have no way of separately identifying \( \eta_i \) from \( \phi_{it} \).\(^{17}\) Overcoming this limitation is rather difficult; we take one particular approach that exploits the panel-feature of our data set. We use the fact that in each sector, \( \varepsilon_{it} \) varies over time but \( \eta_i \) is assumed to be time-invariant; the latter assumption seems reasonable at least within the short time span of the financial crisis. We then construct the entire set of values for \( \eta_i \) that is consistent with both the time-variation in our sample and this assumption.

To do so, we denote \( \bar{\eta}_i \equiv \max_t \varepsilon_{it} \) as the greatest value of the expenditure-to-sales ratio observed for sector \( i \) in our sample period. Provided that we restrict \( \phi_{it} \in (0, 1] \), \( \eta_i \) cannot be lower than \( \bar{\eta}_i \).\(^{18}\) Therefore, \( \eta_i \) must lie in the closed set

\[ \eta_i \in [\bar{\eta}_i, 1], \]  

where the upper bound is 1, as in the case of CRS technology and no other primary factors.

For all subsequent quantitative analysis we provide results for the two extremes: \( \eta_i = \bar{\eta}_i \) and \( \eta_i = 1 \). We refer to the former as the lowest possible decreasing returns-to-scale (DRS) model and to the latter as the constant returns-to-scale (CRS) model.

\(^{17}\)Concerning this particular issue Jones (2013) writes, “there is a fundamental identification problem: we see data on observed intermediate good shares and we do not know how to decompose that data into distortions and differences in technologies.”

\(^{18}\)By setting \( \bar{\eta}_i \) to the maximum observation of (25), we implicitly assume that there is at least one year in our sample in which sector \( i \) faces no distortions; in this year, \( \phi_{it} = 1 \). In all other years, \( \phi_{it} < 1 \). Clearly it could also be the case that the sector always faces distortions in all periods, that is, \( \phi_{it} < 1 \) in all years \( t \); in this case \( \eta_i \) would be bounded away from \( \bar{\eta}_i \) and would fall in the interior of the set in (26).
Correlation of the GZ Measure & Expenditure-to-Sales ratios. As we said, the expenditure-to-sales ratio in (25) varies over time for each sector in our sample. Through the lens of the model and provided that $\eta_i$ remains constant, any time-variation in $\varepsilon_{it}$ is a direct consequence of time-variation in the sectoral wedge, $\phi_{it}$. To the extent that financial frictions are only one out of a myriad of all possible distortions, variation in the GZ distortions constructed in (23) cannot possibly capture all variation in expenditure-to-sales ratios seen in the data. However, one may ask how much of the observed time-series variation it does capture.

To answer this, we first compute the time-series correlation between $\Delta \log \phi_{it}$ and $\Delta \log \varepsilon_{it}$ within each sector. We find that typically within a sector there is positive co-movement between the log changes in its GZ distortion and log changes in its expenditure-to-sales ratio: the mean and median correlations are 19.5% and 22.5%, respectively.\(^{19}\)

We compute the cross-sectional means of $\Delta \log \phi_{it}$ and $\Delta \log \varepsilon_{it}$ in each year and plot their time series over our sample period in the left panel of Figure 3. The cross sectional mean of $\Delta \log \phi_{it}$ is given by the dark, solid line and the cross-sectional mean of $\Delta \log \varepsilon_{it}$ by the grey, dashed line.

The time series of these two cross-sectional means are highly correlated at 72.6%. That is, average movements in GZ distortions appear to be strongly correlated with average movements in expenditure-to-sales ratios. This fact is rather striking, particularly given that these two measures are computed from completely independent data sources. Furthermore, both time series appear to take a large dip from 2008 to 2009.

Finally, for each year we compute the cross-sectional standard deviations of $\Delta \log \phi_{it}$ and $\Delta \log \varepsilon_{it}$ and plot their time series in the right panel of Figure 3. For this figure we normalize the standard deviations in the first year of our sample to one in order to make the two series comparable. In both recessions in our sample—the Dot-com crash of 2001-2002 and the 2008-2009 Financial Crisis—the cross-sectional standard deviations of these measures appear to increase relative to previous years.

### 6.2 Network Amplification during the Financial Crisis

Our main quantitative analysis focuses on the years surrounding the 2008-2009 Financial Crisis. We ask: what is the role played by the input-output network in amplifying the financial frictions documented by Gilchrist and Zakrajšek (2012) during this crisis?

\(^{19}\)The aforementioned statistic provides the mean and median time-series correlation within sectors. We take a closer look at these time-series at the industry level for eight specific industries in Section 3.3 of the online appendix.

One might also consider the cross-sectional correlation of these measures within a specific time period. We find that the cross-sectional correlation between $\Delta \log \phi_{it}$ and $\Delta \log \varepsilon_{it}$ from 2008 to 2009 is rather low: only 2.1%. For comparison, the cross-sectional correlation between $\Delta \log \phi_{it}$ and $\Delta \log y_{it}$ from 2008 to 2009 is 36.3%. This indicates that much of the cross-sectional heterogeneity in expenditure-to-sales ratios is likely determined by other factors.
The Labor Wedge Network Multiplier. In order to answer this question, we must disentangle the aggregate network effect from the impact of the distortions themselves. We thus compare the predictions of our calibrated network model—calibrated to the U.S. input-output tables—to an otherwise equivalent model without any input-output linkages. We begin by defining what we mean by an economy without input-output linkages.

Definition 4. A pure horizontal economy is one in which there are no intermediate goods: all sectors sell their output directly to the household.

The pure horizontal economy is the polar opposite of the pure vertical economy: it has no input-output linkage and is similar to what are known as “spider” economies in the global value chains literature (Antrás et al., 2017; Antrás and de Gortari, 2017; Baldwin and Venables, 2013).

In the context of our quantitative exercise, we construct the economy’s “equivalent horizontal economy” as follows. First, all consumption shares are kept equal to that in the calibrated network economy. We then remove all input-output linkages by taking the limit as the intermediate input share of expenditures goes to zero in all sectors, or equivalently, as the labor share of expenditures $\alpha_i$ approaches one. With the equivalent horizontal economy so constructed, we define our measure of network amplification as follows.

Definition 5. Take any input-output economy and its equivalent horizontal economy and compute the aggregate effect of a symmetric distortion across all sectors. The “labor wedge network multiplier” is the ratio between the percentage change in the labor wedge generated by the input-output economy and that generated by its equivalent horizontal economy.

Because distortions are symmetric across all sectors and across the two economies, the labor wedge network multiplier is a function of the network structure alone.
One may likewise wish to define the “TFP network multiplier” as the ratio between the percentage change in TFP generated by the input-output economy and that generated by its equivalent horizontal economy. However, if distortions are symmetric then this object is undefined. To see this, consider the following result for any pure horizontal economy.

Lemma 2. In any pure horizontal economy an efficiency wedge arises if and only if distortions are asymmetric.

Proof. See the online appendix, Section 1.5.

If all sectors are equally distorted, then primary factors cannot be misallocated. As a result, there is no loss in productive efficiency. Therefore, with symmetric distortions, TFP is necessarily constant in the equivalent horizontal economy and the denominator in the TFP network multiplier is equal to zero.

Results for symmetric distortions. We simulate the two alternative economies for 2008-2009 and report the results in Table 1. For symmetric distortions, we use the cross-sectional mean of $\Delta \log \phi_{it}$ during this period. The first row in Table 1 reports the model-generated log changes in TFP and the labor wedge from 2008 to 2009 in the equivalent horizontal economy with symmetric distortions. The second row reports the model-generated log changes in TFP and the labor wedge from 2008 to 2009 in the calibrated network economy with symmetric distortions. We report these numbers for both the lowest DRS model as well as the CRS model.

The first result we wish to highlight is the model’s estimate of the labor wedge network multiplier. To calculate this multiplier, we simply divide the number in the second row by that

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20In the generic case of asymmetric distortions, an efficiency wedge does emerge. Asymmetries induce primary factors to move from more distorted sectors to less distorted ones, thereby lowering productive efficiency. See, e.g., the literature on misallocation in horizontal CES economies, Restuccia and Rogerson (2008) and Hsieh and Klenow (2009).

21Section 1.1 of the online appendix provides closed-form solutions for the efficiency wedge and the labor wedge for the CRS Cobb-Douglas economy; Section 6.3 of the online appendix provides the analogous results for the DRS Cobb-Douglas economy. We use these representations to compute the efficiency and labor wedges in our numerical simulations.
in the first row; the result is reported in the fourth row of Table 1. The labor wedge network multiplier is 1.70 in the lowest DRS model and 2.12 in the CRS model.

A visual representation of this network amplification is provided in Figures 4 and 5 for the few years surrounding the crisis: 2006-2010. Figure 4 plots the percentage change in the labor wedge generated by the model for the horizontal economy with symmetric shocks (indicated by the dark, dashed line) and the calibrated US input-output economy with symmetric shocks (the dark, solid line); Figure 5 provides the analogous plots for the model-implied TFP. Each figure presents these series for the lowest DRS model in the left panel, and for the CRS model on the right. The amplification generated by the input-output network can be seen by comparing the calibrated network economy with symmetric shocks (the dark, solid line) to the horizontal economy with symmetric shocks (the dark, dashed line).

Figure 4. The model-generated labor wedge for the horizontal economy with symmetric shocks (dark, dashed), the calibrated network economy with symmetric shocks (dark, solid), and the calibrated network economy with asymmetric shocks (gray, dotted) for 2006-2010. Left panel: lowest DRS model. Right panel: CRS model.

Figure 5. The model-generated efficiency wedge for the horizontal economy with symmetric shocks (dark, dashed), the calibrated network economy with symmetric shocks (dark, solid), and the calibrated network economy with asymmetric shocks (gray, dotted) for 2006-2010. Left panel: lowest DRS model. Right panel: CRS model.
In Figure 5 note that TFP in the equivalent horizontal economy is unresponsive to movement in symmetric distortions; this coincides with the result in Lemma 2. Therefore, the TFP network multiplier of symmetric distortions is infinite.

**Results for asymmetric distortions.** We construct the labor wedge network multiplier with symmetric distortions across sectors in order to isolate the role of the network structure. However, as described in the previous subsection, our panel data on GZ distortions exhibits significant heterogeneity across sectors and across time. Moreover, the cross-sectional dispersion in distortions appears to increase during the crisis; see Figure 3. In our full calibration we thereby allow for the asymmetric distribution of measured GZ distortions.

The third row of Table 1 reports the model-generated log changes in TFP and the labor wedge from 2008 to 2009 in the full, calibrated network economy with asymmetric GZ distortions. Similarly, Figures 4 and 5 plot the log changes in the labor wedge and TFP, respectively, generated by this model for the few years surrounding the crisis (2006-2010); this series is represented by the gray, dotted line in these figures.

The second result we wish to highlight is that the full, calibrated network model with asymmetric GZ distortions generates a quantitatively larger drop in the labor wedge than in TFP during the crisis. From 2008 to 2009, the full model produces a 5.91 percentage fall in the labor wedge in the CRS economy and a 4.93 percentage fall in the labor wedge in the DRS economy. In contrast, the full model produces a .28 percentage fall in the efficiency wedge in the CRS model and a .13 percentage fall in the efficiency wedge in the DRS model during the same period.

The full quantitative model thereby generates movement in the labor wedge that is an order of magnitude larger than the model-generated movement in TFP. This is, in fact, consistent with the theory. Theorem 2 informs us that sectoral distortions have first-order effects on the labor wedge, but only second and higher-order effects on the efficiency wedge. As we have abstracted entirely from productivity shocks in our quantitative analysis, it is no surprise that the GZ distortions alone cannot generate large fluctuations in TFP.

### 6.3 A Back-of-the-Envelope Calculation for the Network Multiplier

Our results indicate that the labor wedge network multiplier during the financial crisis is roughly 2. We now relate this number to a certain back-of-the-envelope calculation derived from the theory. In particular, we use Theorem 2 to obtain a sufficient statistic for the labor wedge network multiplier in the CRS economy.

Assume small and symmetric distortions: \( d \log \phi_i = d \log \bar{\phi} \) for all \( i \in I \). Applying Theorem 2, the labor wedge in the CRS network economy must satisfy, to a first-order:
\[ d \log \Lambda_{IO} = \left[ \sum_{i \in I} \lambda_i \right] d \log \bar{\phi}, \]

where \( \sum_{i \in I} \lambda_i \) is the sum of the Domar weights. Applying the same first-order approximation to the CRS horizontal economy yields:

\[ d \log \Lambda_{hor} = d \log \bar{\phi}. \]

In the absence of intermediate goods, gross sales are necessarily equal to total value added. This implies that the sum of the Domar weights—gross sales over GDP—is equal to 1 in the horizontal economy.

The labor wedge network multiplier may thereby be approximated by the ratio of (27) to (28):

\[ \frac{d \log \Lambda_{IO}}{d \log \Lambda_{hor}} = \sum_{i \in I} \lambda_i = \sum_i p_i y_i \frac{C}{C} \]

In this approximation the symmetric shocks cancel out from the numerator and the denominator and we are left only with the sum of the Domar weights in the network economy.

Therefore, an approximate value for the labor wedge network multiplier is the sum of the Domar weights, which itself is equal to the ratio of gross sales to GDP. In our data this ratio is roughly equal to 2 and thereby coincides with the calibrated model’s estimate.

Finally, we have defined the economy’s “equivalent horizontal economy” as the one in which consumption shares are kept equal to those in the calibrated network economy. Note, however, that the first-order approximation presented in (29) is in fact valid for any horizontal economy, irrespective of its distribution of consumption shares. This is because the sum of the Domar weights in any horizontal economy is equal to 1. That said, the irrelevance of the horizontal economy’s vector of consumption shares for this approximation relies on our assumption of symmetric distortions. Were we instead to consider asymmetric distortions, the cross-sectional covariance between the consumption shares (likewise, the Domar weights) and the sectoral distortions would then matter.


Our main quantitative result is that the labor wedge network multiplier of distortions is approximately 2. This measure tells us that if all sectoral distortions were to increase by 1%, then the labor wedge would increase by roughly 2%. We now show how our multiplier relates

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22To see this clearly, note that the vector of Domar weights in any horizontal economy is equal to its vector of consumption expenditure shares, \( \mathbf{v} \). Applying part (ii) of Theorem 2, we have that with symmetric distortions in the horizontal economy, \( d \log \Lambda_{hor} = \sqrt{1} d \log \bar{\phi} = d \log \bar{\phi} \), for all possible consumption expenditure share vectors \( \mathbf{v} \).
to a familiar but distinct intermediate good multiplier found in the roundabout economies of Basu (1995) and Jones (2011, 2013). All proofs for the results presented in this section can be found in the online appendix, Section 5.

**The efficient roundabout economy.** Consider the following simplified version of the one-sector, roundabout production economies in Basu (1995) and Jones (2011, 2013).

The production function of the single-sector is given by

\[ Y(s) = a(s)X(s)^{1-\alpha}L(s)^{\alpha} \]

where \( Y \) is output, \( X \) denotes intermediate goods, \( L \) is labor, and \( \alpha \) is the labor share. We drop the \( i \) subscript as there is only one sector and we let \( a(s) \) denote the productivity in this single sector. The aggregate resource constraint is given by \( C(s) + X(s) = Y(s) \), where \( C(s) \) is aggregate consumption, i.e. GDP. Let us assume for now that labor supply is fixed.

Consider the competitive equilibrium in which there are no distortions. By the First Welfare Theorem we may solve for the equilibrium by solving the planner’s problem. The planner maximizes consumption subject to the resource constraint. This yields optimality condition

\[ X(s) = (1-\alpha)Y(s) \]

Combining this with the resource constraint implies \( C(s) = \alpha Y(s) \). Substituting these conditions into the production function and solving for GDP per worker yields the following expression for TFP in this economy:

\[ A(s) \equiv \frac{C(s)}{L(s)} = \alpha (1-\alpha)^{\frac{1-\alpha}{\alpha}} a(s)^{\frac{1}{\alpha}}, \quad \forall s \in S. \quad (30) \]

With equation (30), Jones (2011, 2013) notes that the intermediate good multiplier in the simple round-about economy is \( 1/\alpha \):

\[ \frac{d \log A}{d \log a} = \frac{1}{\alpha}, \quad \forall s \in S. \quad (31) \]

Specifically, though, this is the “TFP network multiplier of sectoral productivity” in this economy: this multiplier indicates that if productivity at the sectoral level falls by 1%, then TFP at the aggregate level falls by \( 1/\alpha \)%.

Distortions at the micro level, say, among firms within the sector, create input misallocation and thereby lead to a fall in sectoral productivity (as in Restuccia and Rogerson, 2008; Hsieh and Klenow, 2009). The purpose of the Jones (2011, 2013) exercise is to show how the roundabout production structure amplifies a fall in sectoral productivity by a factor of \( 1/\alpha \).

However, Jones (2013) notes that, “small departures from the optimal allocation of labor have tiny effects on TFP (an application of the envelope theorem), but significant misallocation can have very large effects... In the presence of significant misallocation, a small improvement in the allocation of resources will have a large impact on TFP.” That is, in order for micro distortions to have a large effect on TFP, the economy must be away from efficiency; inefficient economies with significant misallocation are the economies Jones (2011, 2013) has in mind.
The distorted roundabout economy. The multiplier in (31) is the TFP network multiplier of sectoral productivities; it is not the labor wedge network multiplier of sectoral distortions as we have defined it in Definition 5.

To calculate the latter in the roundabout economy, we now allow for endogenous labor supply and for a wedge that distorts the single sector’s input decisions. This firm maximizes profits, \( \phi(s)Y(s) - X(s) - W(s)L(s) \), where \( \phi(s) \) is the sectoral distortion (which acts like a tax) and \( W(s) \) is the real wage. As before we assume all proceeds from the distortion are rebated to the household, so that the resource constraint continues to satisfy \( C(s) + X(s) = Y(s) \). In this economy the equilibrium labor wedge is given by:

\[
\Lambda(s) = \frac{\alpha \phi(s)}{1 - (1 - \alpha) \phi(s)}, \quad \forall s \in S.
\] (32)

To a first-order approximation around efficiency,

\[
\frac{d \log \Lambda}{d \log \phi} = \frac{1}{\alpha}.
\] (33)

Therefore, to a first-order, the labor wedge network multiplier in this economy is \( 1/\alpha \).

Connection between the two multipliers. The two network multipliers presented in (31) and (33) are conceptually different. The former indicates how sectoral productivity affects TFP, while the latter indicates how the sectoral distortion affects the labor wedge. Despite their conceptual differences, in the roundabout production economy these measures are both equal to \( 1/\alpha \) (to a first-order around efficiency).

Why is this the case? From our theoretical results in Section 4, we know that these two measures should roughly coincide not only in the special case of the roundabout economy, but also more generally in any multi-sector, input-output economy with CRS technologies. Theorems 1 and 2 indicate that, up to a first-order around efficiency,

\[
\frac{d \log \Lambda}{d \log a} = \frac{d \log A}{d \log a} = \frac{Y}{C}.
\] (34)

That is, not only do these two measures coincide, but they are also approximately equal to the efficient steady-state ratio of gross sales to GDP.

The final step in this analysis is the multipliers’ structural interpretation. In the efficient allocation of the simple roundabout economy, gross sales over GDP satisfies \( Y(s)/C(s) = 1/\alpha \). Combining this with equation (34) yields the desired intermediate good multiplier result.

However, note that this last step—the multipliers’ structural interpretation—is specific to the roundabout production economy but does not hold more generally. Instead, Theorem 3 indicates that for any generic Cobb-Douglas input-output economy, to a first-order around efficiency:

\[
\frac{d \log \Lambda}{d \log \phi} = \frac{d \log A}{d \log a} = v^'L1 = v^' [I_N - \text{diag} (1 - \alpha) G]^{-1} 1,
\] (35)
where recall that $G$ is the input-output network and $1$ denotes a vector of ones of length $N$. The round-about economy is a special case in which there is only one sector, hence $v$ and $G$ are both scalars equal to 1. Applying equation (35) to this special case, $v'L_1 = (1 - (1 - \alpha))^{-1} = 1/\alpha$.

### 6.5 Remarks

We conclude this section with a few final remarks on the limitations of our analysis.

There exist multiple ways of modeling financial frictions, most notably collateral constraints as in Bernanke and Gertler (1989); Kiyotaki and Moore (1997); Bernanke, Gertler, and Gilchrist (1999) and frictions in financial intermediation as in Gertler and Karadi (2011). Our model is static; we thereby abstract completely from capital and the typical forms of collateral constraints. Our model is instead closer in spirit to the within-period working capital constraints present in Jermann and Quadrini (2012) and Bigio (2015) but with the addition of intermediate good purchases.

In our quantitative illustration, we limit technologies to be Cobb-Douglas, thereby imposing an elasticity of substitution of 1 across all inputs. We furthermore allow for perfect mobility of the primary factor, labor, across all sectors. An input elasticity of substitution lower than one—as estimated in Atalay (2017)—and a less than perfectly mobile labor market would make reallocation of inputs across sectors more sclerotic. In this case, we conjecture that sectoral distortions would have an even more pronounced aggregate effect.

Finally, our simplified model of financial frictions abstracts entirely from how firms obtain credit. Working capital constraints are exogenous in our model, and we allow only for bank credit but no trade credit. We refer the reader to recent work by Altinoglu (2020), Luo (2020), and Reischer (2019) for input-output models with endogenous financial constraints that depend on supplier trade credit. Our model is nested in all three papers as the reduced-form in which these constraints are exogenous. Recent empirical work by Costello (2020) moreover provides direct evidence that suppliers exposed to large, exogenous declines in bank financing pass on these shocks to downstream customers through a reduction in the trade credit.

### 7 Conclusion

This paper studies the macroeconomic effects of sectoral distortions in a static, multi-sector, general equilibrium, input-output model. Sectoral distortions manifest at the aggregate level via two channels: the efficiency wedge (TFP) and the labor wedge.

Near efficiency, sectoral distortions have zero first-order effects on TFP, but non-zero first-order effects on the labor wedge. We show that sufficient statistics for the latter effects are the Domar weights. With Cobb-Douglas technologies, these first-order effects map to familiar measures of network “centrality” as captured by the Leontief inverse matrix. Away from
efficiency, moreover, distortions in bottleneck sectors have large effects on the labor wedge, but little to no effect on TFP. In sum, these results contribute to our formal understanding of the mapping from micro shocks to macroeconomic fluctuations.

In a quantitative application, we calibrate the economy to the U.S. input-output tables and use the Gilchrist and Zakrajšek (2012) excess bond premia as a proxy for sectoral financial frictions. We simulate the aggregate effects of these shocks during the 2008-2009 Financial Crisis and produce the following results.

The calibrated economy produces a labor wedge network multiplier of roughly 2, indicating that the U.S. input-output network generates considerable amplification of financial frictions and other distortions. This estimate is in line with a certain back-of-the-envelope calculation—derived from the theory—which indicates that it should roughly be equal to the steady-state ratio of gross sales over GDP; in the data, this number is approximately 2. The calibrated economy moreover generates a fall in the labor wedge that is an order of magnitude larger than the model-generated fall in TFP. This suggests that relative to the efficiency wedge, the labor wedge is the stronger aggregate channel by which sectoral distortions manifest.

We have used this model to shed light on the aggregate effects of within-period financing frictions. This class of financial frictions—working capital constraints, trade credit disruptions, and the collapse of payments—is relevant not only within the historical context of the 2008-2009 Financial Crisis, but also appears to be a resurging major concern during the 2020 COVID-19 crisis. We hope this work leads to future studies on the network amplification of such frictions.

References


8 Appendix

In this appendix, we first state and solve the household and firms’ problems. We then state and prove an intermediate lemma which does not appear in the main text but offers a complete characterization of the equilibrium allocations in our economy. We then proceed with the proofs of the major results that appear in the main text.

Auxiliary Problems and an Intermediate Lemma

We first state the problems solved by the household and the firms. Consider first the household.

The Household’s Problem. In any state $s$, the household chooses its consumption bundle and labor supply $\{c_1(s), \ldots, c_N(s), L(s)\}$ in order to maximize its utility

$$\max U(C(c_1(s), \ldots, c_N(s))) - V(L(s))$$

subject to its budget constraint

$$\sum_{i \in I} p_i(s)c_i(s) \leq W(s)L(s) + \sum_{i \in I} \left( \int \pi_{i,k}(s)dk + \bar{\pi}_i(s) \right) + T(s),$$

taking the wage and prices of the goods $\{W(s), p_1(s), \ldots, p_N(s)\}$ as given.

Consider now the production side.

The Sectoral Aggregator’s Problem. In any state $s$, the sectoral aggregator firm in sector $i$ chooses output and input bundle, $\{y_i(s), (y_{i,k}(s))_k\}$, in order to maximize profit

$$\max p_i(s)y_i(s) \int p_i(s)y_{i,k}(s)dk$$

subject to CES technology (3), taking all prices $\{p_i(s), (p_{i,k}(s))_k\}$ as given.

The sectoral aggregator firm’s problem is standard; it yields the typical CES demand function

$$y_{i,k}(s) = \left( \frac{p_{i,k}(s)}{p_i(s)} \right)^{-\theta_i} y_i(s), \quad \forall s \in S,$$

where the price index for sector $i$ is given by $p_i(s) = \left( \int p_{i,k}(s)^{1-\theta_i}dk \right)^{1/\theta_i}$.

Finally, consider the problem of the individual, monopolistically-competitive firm.

The Monopolist’s Problem. In any state $s$, the monopolistically-competitive firm $k$ within sector $i$ chooses output and inputs $\{y_{i,k}(s), \ell_{i,k}(s), x_{i,k}(s)\}$ so as to maximize its profit

$$\max_{y_{i,k}(s), \ell_{i,k}(s), x_{i,k}(s)} \left( 1 - \tau_i(s) \right)p_{i,k}(s)y_{i,k}(s) - W(s)\ell_{i,k}(s) - \sum_{j \in I} p_j(s)x_{ij,k}(s)$$

subject to its technology, (1) and (2), and subject to the CES demand for its product (36), taking the wage and intermediate input prices $\{W(s), p_1(s), \ldots, p_N(s)\}$ as given.
Equilibrium characterization. We now state an intermediate lemma which provides a complete characterization of the equilibrium. This lemma is followed immediately by its proof.

Lemma 3. An allocation $\xi$ and a price system $\varrho$ compose an equilibrium if and only if the following four properties hold:

(i) the following household optimality conditions are satisfied:

\[
U'(C(s)) \frac{\partial C(s)}{\partial c_i(s)} \frac{W(s)}{p_i(s)} = V'(L(s)), \quad \forall i \in I, s \in S, \tag{38}
\]

\[
\frac{\partial C(s)}{\partial c_i(s)}|_{s} \frac{\partial C(s)}{\partial c_j(s)}|_{s} = \frac{p_j(s)}{p_i(s)}, \quad \forall i, j \in I, s \in S; \tag{39}
\]

(ii) the following firm optimality conditions are satisfied:

\[
\left(\frac{\theta_i(s) - 1}{\theta_i(s)}\right) (1 - \tau_i(s)) \left( z_i(s) \frac{\partial F_i(s)}{\partial \ell_i(s)} - \frac{W(s)}{p_i(s)} \right) = 0, \quad \forall i \in I, s \in S, \tag{40}
\]

\[
\left(\frac{\theta_i(s) - 1}{\theta_i(s)}\right) (1 - \tau_i(s)) \left( z_i(s) \frac{\partial F_i(s)}{\partial x_i(s)} \frac{\partial G_i(s)}{\partial x_{ij}(s)} \right) - \frac{p_j(s)}{p_i(s)} = 0, \quad \forall i, j \in I, s \in S, \tag{41}
\]

along with the CES demand condition (36);

(iii) the household’s budget constraint is satisfied; and

(iv) markets clear:

\[
y_i(s) = c_i(s) + h_i(s) + \sum_{j \in I} x_{ji}(s), \quad \forall i \in I, s \in S, \quad \text{and} \quad \sum_{i \in I} \ell(s) = L(s), \quad \forall s \in S. \tag{42}
\]

Proof. We first derive the household’s optimality conditions. Following this we derive the firm’s optimality conditions.

Household. Consider the household’s problem; let $\lambda(s)$ be the Lagrange multiplier on the household’s budget constraint in state $s$. The household’s first order conditions for consumption of any good and labor are given by:

\[
U'(C(s)) \frac{\partial C(s)}{\partial c_i(s)} - \lambda(s)p_i(s) = 0 \quad \forall i \in I, s \in S, \tag{43}
\]

\[
-V'(L(s)) + \lambda(s)W(s) = 0 \quad \forall s \in S. \tag{44}
\]

The regularity conditions on preferences ensure that the second-order conditions for a maximum are satisfied.

By combining (43) and (44) we derive the household’s optimality condition stated in (38). By combining (43) for good $i$ and (43) for good $j$ we derive the household’s optimality condition stated in (39).

The Sectoral Aggregator Firm. The sectoral aggregator firm’s problem is straight-forward and results in the CES demand function stated in (36).
The Monopolistically-Competitive Firm. Next consider the monopolistically-competitive firm’s problem. The firm’s problem can be split into an inner and outer problem. In the outer problem, the firm chooses output in order to maximize profits:

$$\pi_{i,k}(s) = (1 - \tau_i(s))p_{i,k}(s)y_{i,k}(s) - mc_{i,k}(s)y_{i,k}(s),$$

subject to the firm’s CES demand function (36), where $mc_{i,k}(s)$ is the firm’s marginal cost of producing goods $y_{i,k}(s)$. Substituting in the firm’s demand function, this problem reduces to

$$\max_{y_{i,k}} \left(1 - \tau_i(s)\right) \left(\frac{y_{i,k}(s)}{y_i(s)}\right)^{-1/\theta_i} p_i(s)y_{i,k}(s) - mc_{i,k}(s)y_{i,k}(s).$$

The CES demand ensures that this objective is concave and yields a maximum. At this maximum, the following first-order condition is satisfied:

$$\frac{\theta_i(s) - 1}{\theta_i(s)} (1 - \tau_i(s)) \left(\frac{y_{i,k}(s)}{y_i(s)}\right)^{-1/\theta_i} p_i(s) = mc_{i,k}(s).$$

By symmetry of all firms within the sector, this implies

$$\frac{\theta_i(s) - 1}{\theta_i(s)} (1 - \tau_i(s)) p_i(s) = mc_i(s).$$

(46)

The inner problem determines the firm’s marginal cost function. In particular, the firm’s inner problem is a cost minimization problem given by

$$mc_i(s)y_i(s) = \min W(s)\ell_i(s) + p(s)'x_i(s),$$

(47)

where we $p(s)$ is the vector of commodity prices, subject to the firm’s production function:

$$y_i(s) = z_i(s) F_i(\ell_i(s), G_i(x_i(s))).$$

This yields first-order conditions

$$mc_i(s) = W(s) \left( z_i(s) \frac{\partial F_i(s)}{\partial \ell_i(s)} \right)^{-1} \text{ and } mc_i(s) = p_j(s) \left( z_i(s) \frac{\partial F_i(s)}{\partial x_i(s)} \frac{\partial G_i(s)}{\partial x_{ij}(s)} \right)^{-1}. \quad (48)$$

The Neoclassical conditions on technologies ensure that the second-order conditions for a minimum in this problem are satisfied.

From now on we drop the subscript $k$ on the firm’s marginal cost. Combining the marginal cost conditions in (48) with (46) we obtain optimality conditions for the firm stated in (40) and (41). The firm and household’s optimality conditions, along with budget constraints and market clearing conditions, constitute an equilibrium. 

□
8.1 Proof of Proposition 1

Necessity. We first prove necessity. First, take equation (40). This may be rewritten as
\[
\frac{\theta_i(s) - 1}{\theta_i(s)} \left( 1 - \tau_i(s) \right) \left( \frac{z_i(s)}{p_i(s)} \right) \frac{\partial F_i(s)}{\partial \ell_i(s)} = \frac{W(s)}{p_i(s)}, \quad \forall i \in I, s \in S.
\]
Combining this with the household's optimality condition (38), we obtain
\[
\frac{\theta_i(s) - 1}{\theta_i(s)} \left( 1 - \tau_i(s) \right) \left( \frac{z_i(s)}{p_i(s)} \right) U'(C(s)) \frac{\partial C(s)}{\partial \ell_i(s)} = V'(L(s)), \quad \forall i \in I, s \in S.
\]
Similarly, take equation (41). This may be rewritten as
\[
\frac{\theta_i(s) - 1}{\theta_i(s)} \left( 1 - \tau_i(s) \right) \left( \frac{z_i(s)}{p_i(s)} \right) \frac{\partial F_i(s)}{\partial x_{ij}(s)} \frac{\partial G_i(s)}{\partial x_{ij}(s)} = \frac{p_j(s)}{p_i(s)}, \quad \forall i, j \in I, s \in S.
\]
Combining this with the household's optimality condition (39), we obtain
\[
\frac{\theta_i(s) - 1}{\theta_i(s)} \left( 1 - \tau_i(s) \right) \left( \frac{z_i(s)}{p_i(s)} \right) \frac{\partial F_i(s)}{\partial x_{ij}(s)} \frac{\partial G_i(s)}{\partial x_{ij}(s)} = \frac{\partial C(s)/\partial c_j(s)}{\partial C(s)/\partial c_i(s)}, \quad \forall i, j \in I, s \in S.
\]
Therefore, the following two sets conditions:

\[
\begin{align*}
V'(L(s)) &= \frac{\theta_i(s) - 1}{\theta_i(s)} \left( 1 - \tau_i(s) \right) U'(C(s)) \frac{\partial C(s)}{\partial \ell_i(s)} \frac{\partial F_i(s)}{\partial \ell_i(s)}, \quad \forall i \in I, s \in S, \\
dC(s)/dc_j(s) &= \frac{\theta_i(s) - 1}{\theta_i(s)} \left( 1 - \tau_i(s) \right) z_i(s) \frac{\partial F_i(s)}{\partial x_{ij}(s)} \frac{\partial G_i(s)}{\partial x_{ij}(s)}, \quad \forall i, j \in I, s \in S,
\end{align*}
\]
along with the resource constraints:

\[
L(s) = \sum_{i \in I} \ell_i(s), \quad \text{and} \quad y_i(s) = c_i(s) + h_i(s) + \sum_{j \in I} x_{ji}(s), \quad \forall i \in I, s \in S,
\]
are necessary conditions of the equilibrium allocation.

Next, consider the resource constraint for sector \( i \) in (51). Using the fact that \( h_i(s) = \delta_i(s) \tau_i(s)y_i(s) \), we may rewrite this resource constraint as

\[
y_i(s) = c_i(s) + \sum_{j \in I} x_{ji}(s), \quad \forall i \in I, s \in S.
\]
We define a function \( q_i(s) \) as follows

\[
q_i(s) \equiv y_i(s) - h_i(s) = (1 - \delta_i(s) \tau_i(s)) y_i(s), \quad \forall i \in I, s \in S.
\]
Substituting this into (52) gives us

\[
q_i(s) = c_i(s) + \sum_{j \in I} x_{ji}(s), \quad \forall i \in I, s \in S,
\]
verifying that the resource constraints in (11) preserves the resource constraints in (51) with the functions \( q_i(s) \) and \( \psi(s) \) as defined in (8) and (7), respectively.

Next, take conditions (49) and (50). Multiplying and dividing the right-hand side of both of these equations by \( \psi_i(s) \) yields:

\[
V'(L(s)) = \frac{\theta_i(s)}{\psi_i(s)} \left( 1 - \frac{\tau_i(s)}{1 - \delta_i(s)} \right) \frac{U'(C(s))}{U'(C(s))} \frac{\partial C(s)}{\partial c_i(s)} \psi_i(s) z_i(s) \frac{\partial F_i(s)}{\partial \ell_i(s)}, \quad \forall i \in I, s \in S, \\
\frac{dC(s)/dc_j(s)}{dC(s)/dc_i(s)} = \frac{\theta_i(s)}{\theta_j(s)} \left( 1 - \frac{\tau_i(s)}{1 - \delta_i(s)} \right) \psi_i(s) z_i(s) \frac{\partial F_i(s)}{\partial x_i(s)} \frac{\partial G_i(s)}{\partial x_{ij}(s)}, \quad \forall i, j \in I, s \in S.
\]

Finally, note that:

\[
\frac{\partial q_i(s)}{\partial \ell_i(s)} = \psi_i(s) z_i(s) \frac{\partial F_i(s)}{\partial \ell_i(s)}, \quad \text{and} \quad \frac{\partial q_i(s)}{\partial c_i(s)} = \psi_i(s) z_i(s) \frac{\partial F_i(s)}{\partial c_i(s)}, \quad \forall i \in I, s \in S.
\]

This verifies that optimality conditions (9) and (10) are equivalent to optimality conditions in (49) and (50) with the function \( \phi(s) \) as defined in (7). This concludes necessity.

**Sufficiency.** Consider now sufficiency. From the argument provided above, equilibrium optimality conditions (9) and (10) are equivalent to equilibrium optimality conditions (49) and (50), respectively. Furthermore, the resource constraints in (11) are equivalent to the resource constraints in (51).

Therefore, take any allocation \( \xi(s) \) that satisfies conditions (49)-(50) along with resource constraints (51). We now prove that there exists a set of strictly positive prices \( \{p_1(s), \ldots, p_N(s)\} \) and a wage rate \( W(s) \) that implement this allocation as an equilibrium.

We construct the set of equilibrium prices as follows. First we set the price of the final good equal to 1 (as in our equilibrium definition). We construct the real wage as follows:

\[
W(s) = \frac{V'(L(s))}{U'(C'(s))}, \quad \forall s \in S, \tag{54}
\]

and we construct each sectoral price as follows:

\[
p_i(s) = \frac{\partial C(s)}{\partial c_i(s)}, \quad \forall i \in I, s \in S. \tag{55}
\]

Note that our constructed real wage and prices are all strictly positive due to strictly positive first derivatives of \( U, V \) and \( C \). The ratio of the wage in (54) to the prices in (55) satisfy the household's optimality conditions in (38). Moreover, the ratio of two sectoral prices are given by:

\[
\frac{p_j(s)}{p_i(s)} = \frac{\partial C(s)/\partial c_j(s)}{\partial C(s)/\partial c_i(s)}, \quad \forall i, j \in I, s \in S, \tag{56}
\]

and thereby satisfy the household's optimality condition in (39).

The wage and prices given in (54) and (55), along with the fact that the allocation satisfies (49), ensure that firm optimality (40) is satisfied. Moreover, the price ratio in (56), along with the fact that the allocation satisfies (50), ensures that firm optimality (41) is satisfied.
Finally, for any sector $i$, let relative firm prices satisfy
\[ \frac{p_{i,k}(s)}{p_i(s)} = \left( \frac{y_{i,k}(s)}{y_i(s)} \right)^{-\frac{1}{\tau_i}}, \quad \forall s \in S. \]

With these prices we satisfy the equilibrium conditions (36) for intermediate good demand. Given that $y_{i,k}(s) = y_i(s)$ for all firms $k$ in sector $i$, this immediately implies that $p_{i,k}(s) = p_i(s)$, for all $k \in [0, 1]$.

Market clearing for each good is ensured by the resource constraints in (51). What remains to be checked is that the household's budget constraint is satisfied. Consider the household budget constraint:
\[ \sum_{i \in I} p_i(s)c_i(s) \leq W(s)L(s) + \sum_{i \in I} \left( \int \pi_{ik}(s)dk + \pi_i(s) \right) + T(s). \]

Substituting in for profits from (37) we have that the household's budget constraint satisfies
\[ \sum_{i \in I} p_i(s)c_i(s) \leq W(s)L(s) + \sum_{i \in I} \left( (1 - \tau_i(s)) p_i(s) y_i(s) - W(s)\ell_i(s) - \sum_{j \in I} p_j(s) x_{ij}(s) \right) + T(s). \]

Using the resource constraint for labor (51), this inequality reduces to
\[ \sum_{i \in I} p_i(s)c_i(s) \leq \sum_{i \in I} (1 - \tau_i(s)) p_i(s) y_i(s) - \sum_{i \in I} \sum_{j \in I} p_j(s) x_{ij}(s) + T(s). \]  

Next, take the resource constraint (51) for good $i$ and multiply both sides by $p_i(s)$; this yields
\[ p_i(s) y_i(s) = p_i(s) c_i(s) + p_i(s) h_i(s) + \sum_{j \in I} p_i(s) x_{ji}(s). \]

Summing the above expression across sectors and combining with condition (57) gives us
\[ \sum_{i \in I} p_i(s) y_i(s) - \sum_{i \in I} p_i(s) h_i(s) - \sum_{i \in I} \sum_{j \in I} p_i(s) x_{ji}(s) \leq \sum_{i \in I} (1 - \tau_i(s)) p_i(s) y_i(s) - \sum_{i \in I} \sum_{j \in I} p_j x_{ij}(s) + T(s). \]

Canceling terms on both sides and rearranging yields
\[ \sum_{i \in I} p_i(s) y_i(s) \leq \sum_{i \in I} (1 - \tau_i(s)) p_i(s) y_i(s) + \sum_{i \in I} p_i(s) h_i(s) + T(s). \]  

Using the fact that tax proceeds wasted and rebated satisfy
\[ \sum_{i \in I} p_i(s) h_i(s) = \sum_{i \in I} \delta_i(s)\tau_i(s) p_i(s) y_i(s) \quad \text{and} \quad T(s) = \sum_{i \in I} (1 - \delta_i(s))\tau_i(s) p_i(s) y_i(s), \]

we may re-express equation (57) as follows:
\[ \sum_{i \in I} p_i(s) y_i(s) \leq \sum_{i \in I} (1 - \tau_i(s)) p_i(s) y_i(s) + \sum_{i \in I} \delta_i(s)\tau_i(s) p_i(s) y_i(s) + \sum_{i \in I} (1 - \delta_i(s))\tau_i(s) p_i(s) y_i(s). \]
Collecting terms we may rewrite this as

\[ \sum_{i \in I} p_i(s) y_i(s) \leq \sum_{i \in I} [1 - \tau_i(s) + \delta_i(s) \tau_i(s) + (1 - \delta_i(s)) \tau_i(s)] p_i(s) y_i(s), \]

which holds with equality. Therefore, the resource constraints in (51) ensure that the household's budget constraint is satisfied. QED.

### 8.2 Proof of Proposition 2

Relative to the baseline economy, the only modification is that firm profits are now given by

\[ \pi_{i,k}(s) = p_{i,k}(s) y_{i,k}(s) - (1 + r_i(s)) [W(s) \ell_i(s) + \sum_{j \in I} p_j(s) x_{ij,k}(s)], \]

where \(1 + r_i(s)\) is the intra-period interest rate. The firm chooses inputs and output in order maximize profits subject to the firm's demand function (36).

Following the proof of Lemma 3, the firm's inner cost-minimization problem remains the same. The firm's outer problem becomes

\[ \max_{y_{i,k}} p_{i,k}(s) y_{i,k}(s) - (1 + r_i(s)) mc_i(s) y_{i,k}(s) \]

subject to CES demand function (36), where \(mc_i(s)\) denotes the firm's marginal cost as characterized in (48). This problem yields the following firm optimality condition:

\[ \left( \frac{\theta_i(s) - 1}{\theta_i(s)} \right) \left( \frac{y_{i,k}(s)}{y_i(s)} \right)^{-1/\theta_i} p_i(s) = (1 + r_i(s)) mc_i(s). \]

Imposing symmetry across all firms within a sector, this condition becomes

\[ \left( \frac{\theta_i(s) - 1}{\theta_i(s)} \right) \frac{1}{1 + r_i(s)} p_i(s) = mc_i(s), \quad \forall i \in I, s \in S. \tag{59} \]

This condition is the financial friction economy's counterpart of condition (46) in the baseline economy. Finally, there are no wasted resources as the household budget constraint includes interest payments accrued to the banking sector. Therefore all parts of the proofs for Propositions 3 and 1 remain unchanged, modulo the redefinitions of \(\phi\) and \(\psi\) in (12). QED.

### 8.3 Proof of Theorem 1

As in Hulten (1978), we consider the efficient economy. Since the first welfare theorem holds, the equilibrium allocation solves the planner's problem. The planner's problem may be split into an inner and outer problem. The outer problem is to maximize utility of the representative household given the aggregate production function; that is,
\[
\max U(C(s)) - V(L(s)) \quad \text{s.t.} \quad C(s) = A(s)L(s).
\]

This yields the intratemporal optimality condition
\[
V'(L(s)) = A(s)U'(C(s)), \quad \forall s \in S,
\]
which coincides with the interatemporal condition in (14), but with the labor wedge equal to one: \( \Lambda(s) = 1 \).

The inner problem is to maximize final good output for given labor supply. We define a function \( \bar{C} : \mathbb{R}_+^{N+1} \to \mathbb{R}_+ \) such that it solves:
\[
\bar{C}(a_1, \ldots, a_N, \bar{L}) = \max C(c_1, \ldots, c_N)
\]
subject to the goods resource constraints,
\[
a_i F_i(\ell_i, G_i(x_i)) = c_i + \sum_{j \in I} x_{ji}, \quad \forall i \in I, s \in S,
\]
and subject to the labor resource constraint \( \sum_{i \in I} \ell_i = \bar{L} \), where aggregate labor is fixed at some arbitrary level \( \bar{L} \).

Homogeneity of degree 1 of \( C \) and of \( F_i, G_i \) for all \( i \in I \), implies that we may write
\[
\bar{C}(a_1, \ldots, a_N, L) = \alpha(a_1, \ldots, a_N)L,
\]
for some function \( \alpha : \mathbb{R}_+^N \to \mathbb{R}_+ \). The mapping from this function to the outer problem is therefore given by:
\[
A(s) = \alpha(a_1(s), a_2(s), \ldots, a_N(s)), \quad \forall s \in S.
\]

The solution to the inner problem follows the proof in Baqaee and Farhi (2019). The Lagrangian for this problem is given by
\[
\mathcal{L} = C(c_1, \ldots, c_N) - \sum_{i \in I} \mu_i \left( c_i + \sum_{j \in I} x_{ji} - a_i F_i(\ell_i, G_i(x_i)) \right) - \lambda \left( \sum_{i \in I} \ell_i - 1 \right)
\]
From the envelope theorem,
\[
\frac{d\bar{C}}{da_i} = \frac{d\mathcal{L}}{da_i} = \mu_i F_i(\ell_i, G_i(x_i)), \quad \forall i \in I.
\]
Therefore,
\[
\frac{d\bar{C}}{da_i} a_i = \mu_i y_i, \quad \forall i \in I.
\]
What remains to be shown is that the Lagrange multiplier \( \mu_i \) is equal to the equilibrium price \( p_i \).
To show this, first note that in the planner’s problem, the ratio of these multipliers equals the household’s marginal rate of substitution between goods:

\[
\frac{\partial C}{\partial c_i} = \frac{\mu_i}{\mu_j}, \quad \forall i, j \in I.
\]

This coincides with the equilibrium condition (56): that is, household’s marginal rate of substitution between goods is equated with the equilibrium price ratio. Therefore, the ratio of multipliers satisfies:

\[
\frac{\mu_i}{\mu_{ij}} = \frac{\partial C}{\partial c_i} = \frac{p_i}{p_j}, \quad \forall i, j \in I.
\]

for any \(i, j\). For any good \(i\), we apply (61) to the ratio of its price to the price of good 1:

\[
\frac{\partial C}{\partial c_i} = \frac{\partial C}{\partial c_1} \frac{p_i}{p_1}, \quad \forall i \in I.
\]

Next, because \(C\) is homogenous of degree 1, we may write it as

\[
C = \sum_{i \in I} \frac{\partial C}{\partial c_i} c_i = \frac{\partial C}{\partial c_1} \sum_{i \in I} p_i c_i.
\]

But note that in equilibrium \(C = \sum_{i \in I} p_i c_i\) as we have normalized the price of the consumption good to 1. Combining this with (63), implies that the price of good 1 satisfies \(p_1 = \partial C / \partial c_1\).

Substituting this into (62) yields \(p_i = \partial C / \partial c_i\), for all \(i \in I\). Finally, combining this with (61), yields the following result:

\[
p_i = \frac{\partial C}{\partial c_i} = \mu_i, \quad \forall i \in I,
\]

as was to be shown. QED.

8.4 Proof of Theorem 2

Part (i) TFP effects. The proof of part (i) is nearly trivial. Let us write the state in terms of the two vectors:

\[
a = (a_1, \ldots, a_N)', \quad \text{and} \quad \phi = (\phi_1, \ldots, \phi_N)',
\]

so that we may separate the efficient productivity shocks from the inefficient distortions. Let \(A(a, \phi)\) denote equilibrium TFP in state \((a, \phi)\).

Consider the following hypothetical problem:

\[
\max_{\phi_1, \ldots, \phi_N} A(a, \phi).
\]

That is, choose the vector of distortions \(\phi\) so that equilibrium TFP is at its greatest. The first order condition of this problem is given by \(dA/d\phi_i = 0\), for all \(i \in I\).
But note that \( A(a, \phi) \) attains its maximum when the economy is efficient: \( A(a, 1) = \max_\phi A(a, \phi) \). Therefore, at the efficient benchmark, the derivative of equilibrium TFP with respect to any distortion satisfies:

\[
\frac{d \log A}{d \log \phi_i} \bigg|_{\phi = 1} = \frac{d A}{d \phi_i} \bigg|_{\phi = 1} = 0, \quad \forall i \in I,
\]
as was to be shown.

**Part (ii) Labor wedge effects.** There are no distortions on the household’s side. As a result, for any state \( s \in S \), the household is always on its first-order condition,

\[
\frac{V'(L(s))}{U'(C(s))} = W(s).
\] (64)

That is, in any state, the household’s marginal rate of substitution between labor and consumption is equated with the real wage.

In our model it is only the production side which may be distorted. Let us define \( \pi_i(s) \equiv 1 - \phi(s) \). Profits of the representative firm \( i \) in state \( s \) may be written as:

\[
\pi_i(s) = (1 - \pi_i(s)) p_i(s) q_i(s) - W(s) \ell_i(s) - \sum_{j \in I} p_j(s) x_{ij}(s).
\] (65)

Rearranging equation (65) and summing up over labor inputs yields:

\[
W(s)L(s) = W(s) \sum_{i \in I} \ell_i(s) = \sum_{i \in I} p_i(s) q_i(s) - \sum_{i \in I} \sum_{j \in I} p_j(s) x_{ij}(s) - \sum_{i \in I} \pi_i(s) p_i(s) q_i(s) - \sum_{i \in I} \pi_i(s).
\]

Using the resource constraint, (53), we may rewrite this as

\[
W(s)L(s) = \sum_{i \in I} p_i(s) \left[ c_i(s) + \sum_{j \in I} x_{ij}(s) \right] - \sum_{i \in I} \sum_{j \in I} p_j(s) x_{ij}(s) - \sum_{i \in I} \pi_i(s) p_i(s) q_i(s) - \sum_{i \in I} \pi_i(s).
\]

Noting that \( \sum_{i \in I} \sum_{j \in I} p_i(s) x_{ij}(s) = \sum_{i \in I} \sum_{j \in I} p_j(s) x_{ij}(s) \), and that aggregate consumption satisfies \( C(s) = \sum_{i \in I} p_i(s) c_i(s) \), the above equation reduces to:

\[
W(s)L(s) = C(s) - \sum_{i \in I} \pi_i(s) p_i(s) q_i(s) - \sum_{i \in I} \pi_i(s).
\]

Factoring out \( C(s) \) from this expression yields:

\[
W(s)L(s) = \left[ 1 - \sum_{i \in I} \pi_i(s) \frac{p_i(s) q_i(s)}{C(s)} - \frac{1}{C(s)} \sum_{i \in I} \pi_i(s) \right] C(s).
\]

Next, letting \( \lambda_i(s) \equiv (p_i(s) y_i(s))/C(s) \) denote the equilibrium Domar weight of sector \( i \) in state \( s \), and recalling that \( q_i(s) = \psi_i(s) y_i(s) \), we may rewrite the above expression as:

\[
W(s) = \left[ 1 - \sum_{i \in I} \pi_i(s) \psi_i(s) \lambda_i(s) - \frac{1}{C(s)} \sum_{i \in I} \pi_i(s) \right] \frac{C(s)}{L(s)}.
\] (66)
With the CRS aggregate production function in (13), \( \frac{dC(s)}{dL(s)} = \frac{C(s)}{L(s)} \). Thus, combining (66) with the household’s optimality condition in (64), we obtain the intratemporal condition:

\[
\frac{V'(L(s))}{U'(C(s))} = W(s) = \Lambda(s) \frac{dC(s)}{dL(s)},
\]

with the labor wedge given by:

\[
\Lambda(s) = 1 - \sum_{i \in I} \tilde{\tau}_i(s) \psi_i(s) \lambda_i(s) - \frac{1}{C(s)} \sum_{i \in I} \pi_i(s).
\]

Alternatively, using the fact that we have defined \( \bar{\tau}_i(s) \equiv 1 - \phi_i(s) \), this becomes

\[
\Lambda(s) = 1 - \sum_{i \in I} (1 - \phi_i(s)) \psi_i(s) \lambda_i(s) - \frac{1}{C(s)} \sum_{i \in I} \pi_i(s). \tag{67}
\]

We now take a first-order approximation; let \( s_0 \) denote the state around which we approximate. For any variable \( x \), let \( \tilde{x} \) denote its log deviation from state \( s_0 \):

\[
\tilde{x}(s) \equiv \log x(s) - \log x(s_0).
\]

We rewrite (67) in terms of log-deviations as follows:

\[
\Lambda(s_0) \exp \tilde{\Lambda}(s) = 1 - \sum_{i \in I} \left( 1 - \phi_i(s_0) \exp \tilde{\phi}_i(s) \right) \left( \psi_i(s_0) \exp \tilde{\psi}_i(s) \right) \left( \lambda_i(s_0) \exp \tilde{\lambda}_i(s) \right) - \frac{1}{C(s_0)} \sum_{i \in I} \pi_i(s_0) \exp \tilde{\pi}_i(s). \tag{68}
\]

Taking a first order Taylor approximation of (68) around state \( s_0 \) yields:

\[
\Lambda(s_0) \tilde{\Lambda}(s) \approx \sum_{i \in I} \phi_i(s_0) \psi_i(s_0) \lambda_i(s_0) \tilde{\phi}_i(s) - \sum_{i \in I} (1 - \phi_i(s_0)) \psi_i(s_0) \lambda_i(s_0) \left[ \tilde{\lambda}_i(s) + \tilde{\psi}_i(s) \right] \tag{69}
\]

\[
+ \frac{1}{C(s_0)} \sum_{i \in I} \pi_i(s_0) \tilde{C}(s) - \frac{1}{C(s_0)} \sum_{i \in I} \pi_i(s_0) \tilde{\pi}_i(s).
\]

Equation (69) holds true as an approximation around any arbitrary state \( s_0 \). But now let us label \( s_0 \) as the efficient steady state as we have defined it in Definition 3. In this case, \( \Lambda(s_0) = 1 \), and \( \phi_i(s_0) = 1 \), \( \psi_i(s_0) = 1 \), and \( \pi_i(s_0) = 0 \), for all \( i \in I \). Therefore, around the efficient steady-state, the expression in (69) reduces to \( \tilde{\Lambda} \approx \sum_{i \in I} \lambda_i \tilde{\phi}_i \), or equivalently,

\[
\log \Lambda \approx \sum_{i \in I} \lambda_i \log \phi_i,
\]

where \( \lambda_i \) is the efficient steady-state Domar weight of sector \( i \). QED.