Redistribution through Markets *

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Abstract

Policymakers frequently use price regulations as a response to inequality in the markets they control. In this paper, we examine the optimal structure of such policies from the perspective of mechanism design. We study a buyer-seller market in which agents have private information about both their valuations for an indivisible object and their marginal utilities for money. The planner seeks a mechanism that maximizes agents’ total utilities, subject to incentive and market-clearing constraints. We uncover the constrained Pareto frontier by identifying the optimal trade-off between allocative efficiency and redistribution. We find that competitive-equilibrium allocation is not always optimal. Instead, when there is substantial inequality across sides of the market, the optimal design uses a tax-like mechanism, introducing a wedge between the buyer and seller prices, and redistributing the resulting surplus to the poorer side of the market via lump-sum payments. When there is significant within-side inequality, meanwhile, it may be optimal to impose price controls even though doing so induces rationing.

Keywords: optimal mechanism design, redistribution, inequality, welfare theorems

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1 Introduction

Policymakers frequently use price regulations as a response to inequality in the markets they control. Local housing authorities, for example, often institute rent control to improve housing access for low-income populations. State governments, meanwhile, use minimum wage laws to address inequality in labor markets. And in the only legal marketplace for kidneys—the one in Iran—there is a legally-regulated price floor in large part because the government is concerned about the welfare of organ donors, who tend to come from low-income households. But to what extent are these sorts of policies the right approach—and if they are, how should they be structured? In this paper, we examine this question from the perspective of optimal mechanism design.¹

Price controls introduce multiple allocative distortions: they drive total trade below the efficient level; moreover, because they necessitate rationing, price controls also mean that some of the agents who participate in trade may not be the most efficient ones. Yet at the same time, price controls can shift surplus to poorer market participants. Additionally, as we highlight here, properly structured price controls can identify poorer individuals through their behavior. Thus a policymaker who cannot observe and redistribute wealth directly may instead opt for carefully constructed price controls—effectively, maximizing the potential of the marketplace itself to serve as a redistributive tool. Our main result shows that optimal redistribution through markets can be obtained through a simple combination of lump-sum transfers and rationing.

Our framework is as follows. There is a market for an indivisible good, with a large number of prospective buyers and sellers. Each agent has quasi-linear preferences and is characterized by a pair of values: a value for the good ($v^K$) and a marginal value for money ($v^M$), the latter of which we think of as capturing the reduced-form consequences of agents’ wealth or, more broadly, social and economic circumstances (we discuss the precise meaning of $v^M$ and the interpretation of our model in Section 1.1).² A market designer chooses a

¹For rent control in housing markets, see, for example, van Dijk (2019) and Diamond et al. (2019). For discussion of minimum wages at the state level, see, for example, Rinz and Voorheis (2018) and the recent report of the National Conference of State Legislatures (2019). For discussion of the price floor in the Iranian kidney market, see Ghods and Savaj (2006) and Akbarpour et al. (2019). Price regulations and controls are also common responses to inequality in pharmaceutical markets (see, e.g., Mrazek (2002)), education (see, e.g., Carneiro et al. (2003), Deming and Walters (2017), Tyler (2019)), and transit (see, e.g., Emmerink et al. (1995), Cohen (2018), and also Wu et al. (2012)).

²Our setup implicitly assumes that the market under consideration is a small enough part of the economy that the gains from trade do not substantially change agents’ wealth levels. In fact, utility can be viewed as approximately quasi-linear from a perspective of a single market when it is one of many markets—the so-called “Marshallian conjecture” demonstrated formally by Vives (1987). More recently, Weretka (2018) showed that quasi-linearity of per-period utility is also justified in infinite-horizon economies when agents are sufficiently patient.
mechanism that allocates both the good and money to maximize the sum of agents’ utilities, subject to market-clearing, budget-balance, and individual-rationality constraints. Crucially, we also require incentive-compatibility: the designer knows the distribution of agents’ characteristics but does not observe individual agents’ values directly; instead, she must infer them through the mechanism. We show that each agent’s behavior is completely characterized by the ratio of her two values $v^K/v^M$, i.e., her rate of substitution. As a result, we can rewrite our two-dimensional mechanism design problem as a unidimensional problem with an objective function equal to the weighted sum of agent’s utilities, with each agent receiving a welfare weight that depends on that agent’s rate of substitution and side of the market.

In principle, mechanisms in our setting can be quite complex, offering a (potentially infinite) menu of prices and quantities (i.e., transaction probabilities) to agents. Nonetheless, we find that there exists an optimal menu with a simple structure. Specifically, we say that a mechanism offers a rationing option if agents on a given side of the market can choose to trade with some strictly interior probability. We prove that the optimal mechanism needs no more than a total of two distinct rationing options on both sides of the market. Moreover, if at the optimum some monetary surplus is generated and passed on as a lump-sum transfer, then at most one rationing option is needed. In this case, one side of the market is offered a single posted price, while the other side can potentially choose between trading at some price with probability 1, or trading at a more attractive price (higher for sellers; lower for buyers) with probability less than 1, with some risk of being rationed.

The simple form of the optimal mechanism stems from our large-market assumption. We notice that any incentive-compatible mechanism can be represented as a pair of lotteries over quantities, one for each side of the market. Hence, the market-clearing constraint reduces to an equal-means constraint—the average quantity sold by sellers must equal the average quantity bought by buyers. It then follows that the optimal value is obtained by concavifying the buyer- and seller-surplus functions at the market-clearing trade volume. Since the concave closure of a one-dimensional function can always be obtained by a binary lottery, we can derive optimal mechanisms that rely on implementing a small number of distinct trading probabilities (i.e., a small number of distinct quantities).\(^3\)

Given our class of optimal mechanisms, we then examine which combinations of lump-sum transfers and rationing are optimal as a function of the characteristics of market participants. We focus on two forms of inequality that can be present in the market. Cross-side inequality measures the average difference between buyers’ and sellers’ values for money, while same-side inequality measures the dispersion in values for money within each side of the market.

\(^3\)The exact intuition is more complicated due to the presence of the budget-balance constraint—see Section 4 for details.
We find that cross-side inequality determines the direction of the lump-sum payments—the surplus is redistributed to the side of the market with a higher average value for money—while same-side inequality determines the use of rationing.

Concretely, under certain regularity conditions, we prove the following results: When same-side inequality is not too large, the optimal mechanism is “competitive,” that is, it offers a single posted price to each side of the market and clears the market without relying on rationing. Even so, however, the designer may impose a wedge between the buyer and seller prices, redistributing the resulting surplus as a lump-sum transfer to the “poorer” side of the market. The degree of cross-side inequality determines the magnitude of the wedge—and hence determines the size of the lump-sum transfer. When same-side inequality is substantial, meanwhile, the optimal mechanism may offer non-competitive prices and rely on rationing to clear the market. Finally, there is an asymmetry in the way rationing is used on the buyer and seller sides—a consequence of a simple observation that, everything else being equal, the decision to trade identifies sellers with the lowest ratio of $v^K$ to $v^M$ (that is, “poorer” sellers, with a relatively high $v^M$ in expectation) and buyers with the highest ratio of $v^K$ to $v^M$ (that is, “richer” buyers, with a relatively low $v^M$ in expectation).

On the seller side, rationing allows the designer to reach the “poorest” sellers by raising the price that those sellers receive (conditional on trade) above the market-clearing level. In such cases, the designer uses the redistributive power of the market directly: willingness to sell at a given price can be used to identify—and effectively subsidize—sellers with relatively higher values for money. Rationing in this way is socially optimal when (and only when) it is the poorest sellers that trade, i.e., when the volume of trade is sufficiently small. This happens, for example, in markets where there are relatively few buyers. Often, the optimal mechanism on the seller side takes a simple form of a single price raised above the market-clearing level.

By contrast, at any given price, the decision to trade identifies buyers with relatively lower values for money. Therefore, unlike in the seller case, it is never optimal to have buyer-side rationing at a single price; instead, if rationing is optimal, the designer must offer at least two prices: a high price at which trade happens for sure and that attracts buyers with high willingness to pay (such buyers are richer on average) and a low price with rationing at which poorer buyers may wish to purchase. Choosing the lower price identifies a buyer as poor; the market then effectively subsidizes that buyer by providing the good at a low price (possibly 0) with positive probability. Using the redistributive power of market for buyers, then, is only possible when sufficiently many (rich) buyers choose the high price, so that the low price attracts only the very poorest buyers. In particular, for rationing on buyer side to make sense, the volume of trade must be sufficiently high. We show that there
are markets in which having a high volume of trade, and hence buyer rationing, is always suboptimal, regardless of the imbalance in the sizes of the sides of the market. In fact, we argue that—in contrast to the seller case—buyer rationing can only become optimal under relatively narrow circumstances.

Our results may help explain the widespread use of price controls and other market-distorting regulations in settings with inequality. Philosophers (e.g., Satz (2010), Sandel (2012)) and policymakers (see, e.g., Roth (2007)) often speak of markets as having the power to “exploit” participants through prices. The possibility that prices could somehow take advantage of individuals who act according to revealed preference seems fundamentally unnatural to an economist. Yet our framework illustrates at least one sense in which the idea has a precise economic meaning: as inequality increases and induces a stronger desire of the designer to redistribute, setting prices competitively becomes dominated by mechanisms that may involve non-market features such as lump-sum transfers and rationing. At the same time, however, our approach suggests that the proper social response to this problem is not banning or eliminating markets—as Sandel (2012) and others suggest—but rather designing market-clearing mechanisms in ways that directly attend to inequality. Policymakers can “redistribute through the market” by choosing market-clearing mechanisms that give up some allocative efficiency in exchange for increased equity.

We emphasize that it is not the point of this paper to argue that markets are a superior tool for redistribution relative to more standard approaches that work through the tax system. Rather, we think of our “market design” approach to redistribution as complementary to public finance at the central government level: Indeed, many local regulators are responsible for addressing inequality in individual markets, without access to macro-economic policy tools; our framework helps us understand how those regulators should set policy. Conceptually, we are thus asking a different question from much of public finance—we seek to understand the equity–efficiency trade-off in market-clearing, with agents exchanging an indivisible good. The redistribution question in our context is in some ways simpler to analyze because of structure imposed by our market design focus. Most notably, because goods in our setting are indivisible and agents have linear utility with unit demand, we find that the optimal mechanism takes a particularly simple form that allows us to assess how the qualitative features of the mechanism—such as rationing and lump-sum transfers—depend on the type and degree of inequality in the market.

At the same time, our approach imposes a number of restrictions that are absent from much of public finance. First and foremost, we take inequality as given: formally, our welfare

4That said, our framework is especially related to the frameworks of Scheuer (2014) and Scheuer and Werning (2017), as we discuss in Section 1.2.
weights are determined exogenously (from agents’ joint distributions of values for the good and money), whereas in public finance those weights can often be determined endogenously through the equilibrium income distribution. Additionally, agent types in our model reflect valuations for the good, rather than productivity or ability—and the good agents trade in our setting is homogenous, whereas many public finance settings can allow differences in skill that make agents imperfectly substitutable. Lastly, our assumption of unit demand with linear utility might be appropriate for studying behavior in a single market but would be too limiting in a standard public finance setting. Public finance models typically assume concavity of the utility function and rely on first-order conditions to characterize agents’ behavior. Agents’ behavior in our model is instead described by a bang-bang solution; this underlies the simple structure of our optimal mechanism because it limits the amount of information that the designer can infer about agents from their equilibrium behavior.

The remainder of this paper is organized as follows. Section 1.1 explains how our approach relates to the classical mechanism design framework and welfare theorems. Section 1.2 reviews the related literature in mechanism design, public finance, and other areas. Section 2 lays out our framework. Then, Section 3 works through a simple application of our general framework, building up the main intuitions and terminology by starting with simple mechanisms and one-sided markets. In Section 4 and Section 5, we identify optimal mechanisms in the general case, and then examine how our optimal mechanisms depend on the level and type of inequality in the market. Section 6 discusses policy implications; Section 7 concludes.

1.1 Interpretation of the model and relation to welfare theorems

Two important consequences of wealth distribution for market design are that (i) individuals’ preferences may vary with their wealth levels, and (ii) social preferences may naturally depend on individuals’ wealth levels (typically, with more weight given to less wealthy or otherwise disadvantaged individuals). The canonical model of mechanism design with transfers assumes that individuals have quasi-linear preferences—ruling out wealth’s consequence (i) for individual preferences. Moreover, in a less obvious way, quasi-linearity along with the Pareto optimality criterion also rule out wealth’s consequence (ii) for social preferences, by implying that any monetary transfer between agents is neutral from the point of view of the designer’s objective (utility is perfectly transferable). In this way, the canonical framework fully separates the question of maximizing total surplus from distributional concerns—all that matters are the agents’ rates of substitution between the good and money, conventionally referred to as agents’ values.

Our work exploits the observation that while quasi-linearity of individual preferences
(consequence (i)) is key for tractability, the assumption of perfectly transferable utility (consequence (ii)) can be relaxed. By endowing agents with two-dimensional values \((v^K, v^M)\), we keep the structure of individual preferences the same while allowing the designer’s preferences to depend on the distribution of money among agents. In our framework, the rate of substitution \(v^K/v^M\) still describes individual preferences, while the “value for money” \(v^M\) measures the contribution to social welfare of transferring a unit of money to a given agent—it is the “social” value of money for that agent, which could depend on that agent’s monetary wealth, social circumstances, or status.

In that sense, our marginal values for money serve the role of Pareto weights—we make this analogy precise in Appendix A.1, where we show a formal equivalence between our two-dimensional value model and a standard quasi-linear model with one-dimensional types and explicit Pareto weights. The idea of using values for money as a measure of social preferences has been already applied in the public finance literature: Saez and Stantcheva (2016), for example, introduced generalized social marginal welfare weights in the context of optimal tax theory and interpreted them as the value that society puts on providing an additional dollar of consumption to any given individual.

Of course, when the designer seeks a market mechanism to maximize a weighted sum of agents’ utilities, an economist’s natural response is to think about welfare theorems. The first welfare theorem guarantees that we can achieve a Pareto-optimal outcome by implementing the competitive-equilibrium mechanism. The second welfare theorem predicts that we can moreover achieve any split of surplus among the agents by redistributing endowments prior to trading (which in our simple model would just take the form of redistributing money holdings). Thus, the argument would go, allowing for Pareto weights in the designer’s objective function should not create a need to adjust the market-clearing rule—competitive pricing should remain optimal. This argument does not work, however, when the designer faces incentive-compatibility (IC) and individual-rationality (IR) constraints. Indeed, while the competitive-equilibrium mechanism is feasible in our setting, redistribution of endowments is not: It would in general violate both IR constraints (if the designer took more from an agent than the surplus that agent appropriates by trading) and IC constraints (agents would not reveal their values truthfully if they expected the designer to decrease their monetary holdings prior to trading). This point is illustrated in Figure 1.1, which depicts the Pareto frontier that would be feasible in a marketplace if the designer could directly observe agents’ values and did not face participation constraints (blue curve). As expected, the unconstrained Pareto frontier is a line because agents’ preferences are quasi-linear. By contrast, however, the constrained Pareto frontier that the designer can achieve in the presence of IC and IR constraints (red curve) is concave and coincides with the unconstrained frontier only
Concavity of the (constrained) Pareto frontier means that IC and IR introduce a trade-off between efficiency and redistribution, violating the conclusion of the second welfare theorem. For example, giving sellers more surplus than in competitive equilibrium requires raising additional revenue from the buyers which—given our IC and IR constraints—can only be achieved by limiting supply. Understanding this efficiency–equity trade-off in the context of marketplace design is the subject of our paper. In particular, we characterize the canonical class of mechanisms that generate the constrained Pareto frontier and show that competitive-equilibrium is suboptimal when the designer has sufficiently strong redistributive preferences.\(^5\)

1.2 Related work

It is well-known in economics (as well as in the public discourse) that a form of price control (e.g., a minimal wage) can be welfare-enhancing if the social planner has a preference for redistribution. That observation was made in the theory literature at least as early as

\(^5\)In analyzing an antitrust setting with “countervailing power,” Loertscher and Marx (2019) examine a mechanism design framework with heterogeneous bargaining weights, building off results of Williams (1987). Because of the heterogeneous weights and incentive constraints in their setting, Loertscher and Marx (2019) identify a similarly-shaped frontier to the one we find here.
Weitzman (1977), who showed that a fully random allocation (an extreme form of price control corresponding to setting a wage at which all workers want to work) can be better than competitive pricing (a “market” wage) when the designer cares about redistribution.

The question of whether optimal taxation should be supplemented by market rationing has since been examined in the optimal taxation literature. Guesnerie and Roberts (1984), for instance, investigated the desirability of anonymous quotas (i.e., quantity control and subsequent rationing) when only linear taxation is feasible; they showed that when the social cost of a commodity is different from the price that consumers face, small quotas around the optimal consumption level can improve welfare. Guesnerie and Roberts (1984) focus particularly on linear taxation.

For the case of labor markets specifically, Allen (1987), Guesnerie and Roberts (1987), and Boadway and Cuff (2001) have shown that, with linear taxation, some form of minimum wage can be welfare-improving. Both Allen (1987, Section IV) and Guesnerie and Roberts (1987, Section 4) also study models with two types of workers (high- and low-skilled) and investigate the desirability of minimum wages when non-linear taxation is available. Allen (1987) and Guesnerie and Roberts (1987) find that that minimum wages are generically suboptimal under non-linear taxation, as they strengthen the binding incentive constraint that prevents high-skilled workers from mimicking low-skilled ones. A critique of that work, however, is the assumed observability of hourly wages, as discussed by Cahuc and Laroque (2014). If the hourly wage is observable—which is necessary for implementing the minimum wage policy—then the government should be able to impose a tax schedule that depends on income and on the hourly wage. The papers just described mostly study the efficiency of the minimum wage under the assumption of perfect competition. Cahuc and Laroque (2014), on the other hand, considered a monopolistic labor market in which firms set the wages; even there, for empirically relevant settings, Cahuc and Laroque (2014) found minimum wages to be suboptimal.

Lee and Saez (2012), meanwhile, showed that minimum wages can be welfare-improving—even when they reduce employment on the extensive margin—so long as rationing is efficient, in the sense that those workers whose employment contributes the least to social surplus leave the market first. At the same time, Lee and Saez (2012) found that minimum wages are never optimal in their setting if rationing is uniform—yet as our analysis highlights, that conclusion derives in part from the fact that Lee and Saez (2012) looked only at a small first-order perturbation around the equilibrium wage. Indeed, our results identify a channel through which even uniform rationing can be optimal. More precisely, when rationing becomes significant (as opposed to a small perturbation around the equilibrium), it influences the incentives of agents to sort into different choices (in our setting, no trade,
rationing, or trade at competitive price). Thus, in our setting, endogenous sorting allows the planner to identify the poorest traders through their behavior.

Moreover, our results give some guidance as to when rationing is optimal: In the setting of Lee and Saez (2012), the inefficiency of rationing is second-order because Lee and Saez (2012) assumed efficient sorting; this is why rationing in the Lee and Saez (2012) model is always optimal. In our setting, we use uniform rationing, which creates a first-order inefficiency; this is why need sufficiently high same-side inequality before we can justify rationing.

Our paper is also related to studies of price controls as a redistributive tool. Viscusi et al. (2005) discussed “allocative costs” of price regulations, and Bulow and Klemperer (2012) characterized when price controls can be harmful to all market participants. In the same vein, our paper also relates to empirical work that seeks to quantify the welfare costs of price regulations. For instance, Glaeser and Luttmer (2003) quantified the costs of allocative inefficiency of rent control in New York City. Autor et al. (2014) and Diamond et al. (2019), meanwhile, studied rent control policies in Boston and San Francisco, respectively. Autor et al. (2014) found that eliminating rent control led to price appreciation. Diamond et al. (2019), on the other hand, found that rent control improved current tenant welfare in the short-run, but also reduced housing supply—and thus Diamond et al. (2019) concluded that rent control is likely to increase prices in the long-run.6

Meanwhile, the idea of using nonuniform welfare weights is a classic idea in public finance (see, e.g., Diamond and Mirrlees (1971), Atkinson and Stiglitz (1976), Saez and Stantcheva (2016)). We bring the nonuniform welfare weights approach from public finance into a complementary mechanism design framework, and are able to fully compute the optimal mechanism. In our setting, the designer has to elicit information about which sellers are poorest through the mechanism. This approach pushes in favor of rationing—even if we are restricted to uniform rationing, and even when non-linear taxation is possible—because it helps identify the poorest sellers through sorting. However, the value of sorting outweighs rationing’s allocative inefficiency only when dispersion in welfare weights is sufficiently high, which in our model corresponds to high same-side inequality.

Additionally, we obtain a different welfare weight structure from many public finance models. Indeed, in public finance, welfare weights are often smaller (in equilibrium) for individuals who “transact” more (e.g., those who provide more high-quality labor). In our setting, because welfare weights are assumed non-increasing in the willingness to pay for the good, whether they are decreasing or increasing in an individual’s equilibrium volume of trade depends on that individual’s side of the market. For the seller side, the direction

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6Similarly, we have some connection to the empirical work on minimum wages; see Dube (2019) for a recent survey.
is reversed compared to a standard public finance model, which is what makes rationing a particularly useful instrument on that side of the market.

Our principal divergence from classical market models—the introduction of heterogeneity in marginal values for money—has a number of antecedents outside of public finance, as well. Condorelli (2013), for example, asks a question similar to ours, working in an object allocation setting in which agents’ willingness to pay is not necessarily the characteristic that appears in the designer’s objective. Condorelli (2013) provides conditions for optimality of non-market mechanisms in his setting. Although our framework is different across several dimensions, the techniques Condorelli (2013) employed to handle ironing in his optimal mechanism share kinship with how we use concavification to solve our problem.7

Huesmann (2017) studies the problem of allocating an indivisible item to a mass of agents, in which agents have different wealth levels, and non-quasi-linear preferences. Esteban and Ray (2006) consider a model of lobbying under inequality in which, similarly to our setting, it is effectively more expensive for less wealthy agents to spend resources in lobbying. More broadly, the idea that it is more costly for low-income individuals to spend money derives from capital market imperfections that impose borrowing constraints on low-wealth individuals; such constraints are ubiquitous throughout economics (see, e.g., Loury (1981), Aghion and Bolton (1997), McKinnon (2010)). Subsequent to our work, and building on some of our ideas, Kang and Zheng (2019) characterize the set of constrained Pareto optimal mechanisms for allocating one good and one bad to a finite set of asymmetric agents, with each agent’s role—a buyer or a seller—determined endogenously by the mechanism.

The idea of using public provision of goods as a form of redistribution (which is inefficient from an optimal taxation perspective) has also been examined (see, e.g., Besley and Coate (1991), Blackorby and Donaldson (1988), Gahvari and Mattos (2007)). Hendren (2017) estimated efficient welfare weights (accounting for the distortionary effects of taxation), and concluded that surplus to the poor should be weighted up to twice as much as surplus to the rich.

Unlike our work—which considers a two-sided market in which buyers and sellers trade—both optimal allocation and public finance settings typically consider efficiency, fairness, and other design goals in single-sided market contexts. Additionally, our work specifically complements the broad literature on optimal taxation by considering mechanisms for settings in which global redistribution of wealth is infeasible, and the designer must respect a participation constraint. For comparison, see for example the work of Stantcheva (2014), who

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7Meanwhile, Loertscher and Muir (2019) use related tools to provide a complementary argument for why non-competitive pricing may arise in practice—showing that in private markets, non-competitive pricing may be the optimal behavior of a monopolist seller at any quantity for which the revenue function is convex, so long as resale can be prevented.
solved for the optimal tax scheme in the model of Miyazaki (1977), or recent papers on tax incidence such as those of Rothschild and Scheuer (2013) or Sachs et al. (2017).

More similar to our work here is that of Scheuer (2014) and Scheuer and Werning (2017), who studied taxation in a “two-sided” market: In the setting of Scheuer (2014), agents have two-dimensional types (like in our model), but the dimensions represent a baseline skill level and a taste for entrepreneurship. An agent’s type affects her occupational choice on both extensive and intensive margins. More precisely, all individuals are ex ante the same and—after the realization of their private types—they can choose whether to be entrepreneurs or workers. In our setting, buyers and sellers are identifiable ex ante, and their choice is whether to trade. The challenge of Scheuer (2014) is that income distributions of workers and entrepreneurs have overlapping supports: high-skilled agents may remain workers due to high costs of entrepreneurship, and low-skilled workers may enter entrepreneurship because they have low costs of doing so. We face a similar challenge, but independently for each side of the market: high-value buyers may choose not to buy because of high marginal utility for money, and low-value buyers may choose to buy because of low value for money (and similarly for sellers). Scheuer (2014) proved that the optimal tax schedules faced by workers and entrepreneurs are different; this resembles our finding that buyers and sellers may face different prices. Scheuer and Werning (2017), meanwhile, studied an assignment model in which firms decide how much labor to demand as a function of their productivity levels, and workers decide how much labor to provide depending on their ability, all by solving their relevant first-order conditions.

Despite these similarities, our work is substantively different from Scheuer (2014) and Scheuer and Werning (2017) both technically and conceptually. Perhaps most importantly, because of our market design focus, buyers and sellers in our model can trade exactly one unit each and have linear utilities. Consequently, the solution to an individual’s problem is not interior, and thus we cannot employ a standard first-order condition approach to characterize the individuals’ responses to the mechanism. Moreover, the optimal mechanism often features bunching, which is explicitly ruled out by Scheuer (2014)—in fact, bunching (rationing) is a key feature of the optimal mechanism that we focus on. Because of that, we are forced to develop different methods, and the Scheuer (2014) results do not extend to our setting. The Scheuer and Werning (2017) model, in addition, exhibits super-modularity in the assignment, which leads to Becker-style assortative matching and makes the economics of the problem quite different from ours. Finally, as we are concerned with market design applications, our model includes participation constraints, which are typically not imposed in optimal taxation models, including those of Scheuer (2014) and Scheuer and Werning (2017).
Our modeling technique, and in particular the inclusion of two-dimensional types, also bears some resemblance to the design problem of two-sided matching markets considered by Gomes and Pavan (2016, 2018). In the Gomes and Pavan setting, agents differ in two dimensions that have distinct influence on match utilities; Gomes and Pavan study conditions on the primitives under which the welfare- and profit-maximizing mechanisms induce a certain simple matching structure. Our analysis differs both in terms of the research question (we focus on wealth inequality) and the details of the model (we include a budget constraint, do not consider matching between agents, and allow for general Pareto weights).

Laffont and Robert (1996), Che and Gale (1998), Fernandez and Gali (1999), Che and Gale (2000), Che et al. (2012), Dobzinski et al. (2012), Pai and Vohra (2014), and Kotowski (2017) analyze allocation problems with budget constraints, which can be seen as an alternative way of modeling the allocative consequences of wealth disparities. While the literature on budget constraints sometimes identifies similar solutions to those we find here (in particular, rationing), that work studies a fundamentally different question. Indeed, the work on budget constraints is interested in how constraints affect allocative efficiency rather than the possibility of redistribution. In our setting, unlike in settings with budget constraints, first-best allocative efficiency is always feasible; hence, our reasons for arriving at rationing are different. Our work also connects to mechanism design models with non-standard agent utility—for example those with non-linear preferences (see, e.g., Maskin and Riley (1984), Baisa (2017)), or ordinal preferences/non-transferable utility (see, e.g., Gale and Shapley (1962), Roth (1984), Hatfield and Milgrom (2005)).

We find that suitably designed market mechanisms (if we may stretch the term slightly beyond its standard usage) can themselves be used as redistributive tools. In this light, our work also has kinship with the broad and growing literature within market design that shows how variants of market mechanisms can achieve fairness and other distributional goals in settings that (unlike ours) do not allow transfers (see, e.g., Hylland and Zeckhauser (1979), Bogomolnaia and Moulin (2001), Budish (2011), Prendergast (2017)). Finally, our work is related to that of Akbarpour and van Dijk (2017), who model wealth inequality as producing asymmetric access to private schools, and show that this changes some welfare conclusions of the canonical school choice matching models.

2 Framework

We study a two-sided buyer-seller market with inequality. There is a unit mass of owners, and a mass $\mu$ of non-owners in the market for a good $K$. All agents can hold at most one unit of $K$ but can hold an arbitrary amount of money $M$. Owners possess one unit of good
Each agent has values $v^K$ and $v^M$ for units of $K$ and $M$, respectively. If $(x^K, x^M)$ denotes the holdings of $K$ and $M$, then an agent with type $(v^K, v^M)$ receives utility

$$v^Kx^K + v^Mx^M.$$ 

The pair $(v^K, v^M)$ is distributed according to a joint distribution $F_S(v^K, v^M)$ for sellers, and $F_B(v^K, v^M)$ for buyers. The designer knows the distribution of $(v^K, v^M)$ on both sides of the market, and can identify whether an agent is a buyer or a seller, but does not observe individual realizations of values.

The designer is utilitarian and aims to maximize the total expected utility from allocating both the good and money. The designer selects a trading mechanism that is “feasible,” in the sense that it satisfies incentive-compatibility, individual-rationality, budget-balance, and market-clearing constraints. (We formalize the precise meaning of these terms in our context soon; we also impose additional constraints in Section 3, which we subsequently relax.)

We interpret the parameter $v^M$ as representing the marginal utility that society (as reflected by the designer) attaches to giving an additional unit of money to a given agent. We refer to agents with high $v^M$ as being “poor.” The interpretation is that such agents have a higher marginal utility of money because of their lower wealth or adverse social circumstances. Analogously, we refer to agents with low $v^M$ as “rich” (or “wealthy”). Heterogeneity in $v^M$ implies that utility is not fully transferable. Indeed, transferring a unit of $M$ from an agent with $v^M = 2$ to an agent with $v^M = 5$ increases total welfare (the designer’s objective) by 3. This is in contrast to how money is treated in a standard mechanism design framework that assumes fully transferable utility: When all agents value good $M$ equally, the allocation of money is irrelevant for total welfare.

In a market in which good $K$ can be exchanged for money, a parameter that fully describes the behavior of any individual agent is the marginal rate of substitution $r$ between $K$ and $M$, that is, $r = v^K/v^M$. This is a consequence of the basic fact that rescaling the utility of any agent does not alter his or her preferences: The behavior of an agent with values $(10, 1)$ does not differ from the behavior of an agent with values $(20, 2)$. As a consequence, by observing agents’ behavior in the market, the designer can at most hope to infer agents’ rates of substitution.\footnote{This claim is nonobvious when arbitrary mechanisms are allowed—but in Section 4 we demonstrate that there is a formal sense in which the conclusion holds.} We denote by $G_j(r)$ the cumulative distribution function of the rate of substitution induced by the joint distribution $F_j(v^K, v^M)$, for $j \in \{B, S\}$. We let $\underline{r}_j$ and $\bar{r}_j$ denote the lowest and the highest $r$ in the support of $G_j$, respectively. Unless
stated otherwise, we assume throughout that the equation \( \mu(1 - G_B(r)) = G_S(r) \) has a unique solution, implying existence and uniqueness of a competitive equilibrium with strictly positive volume of trade.

Even though the designer cannot learn \( v^K \) and \( v^M \) separately, the rate of substitution is informative about both parameters. In particular, fixing the value for the good \( v^K \), a buyer with higher willingness to pay \( r = v^K/v^M \) must have a lower value for money \( v^M \); consequently, the correlation between \( r \) and \( v^M \) may naturally be negative. For example, under many distributions, a buyer with willingness to pay 10 is more likely to have a low \( v^M \) than a buyer with willingness to pay 5. In this case, our designer will value giving a unit of money to a trader with rate of substitution 5 more than to a trader with rate of substitution 10. To see this formally, we observe that the designer’s preferences depend on the rate of substitution \( r \) through two terms: the (normalized) utility, which we denote \( U_B(r) \), and the expected value for money conditional on \( r \), which we denote \( \lambda_B(r) \). Indeed, the expected contribution of a buyer with allocation \((x^K, x^M)\) to the designer’s objective function can be written as

\[
\mathbb{E}^B_{(v^K, v^M)}[v^K x^K + v^M x^M] = \mathbb{E}^B_{(v^K, v^M)}[v^M \left[ \frac{v^K}{v^M} x^K + x^M \right]] \\
= \mathbb{E}^B_r \left[ \mathbb{E}^B_{\{v^M \mid r\}}[r x^K + x^M] \right]. \tag{2.1}
\]

Equality (2.1) also allows us to reinterpret the problem as one where the designer maximizes a standard utilitarian welfare function with Pareto weights \( \lambda_j(r) \) equal to the expected value for money conditional on a given rate of substitution \( r \) on side \( j \) of the market:

\[ \lambda_j(r) = \mathbb{E}^j]\left[ v^M \mid \frac{v^K}{v^M} = r \right] \]

(see Appendix A.1 for further details). This highlights the difference between our model and the canonical mechanism design framework, which implicitly assumes \( \lambda_j(r) = 1 \) for all \( j \) and \( r \). In both settings, \( r \) determines the behavior of agents; but in our model \( r \) also provides information that the designer can use to weight agents’ utilities in the social objective.

### 3 Simple mechanisms

In this section, we work through a simple application of our general framework, building intuitions and terminology that are useful for the full treatment we give in Section 4. In order
to highlight the economic insights, in this section we impose two major simplifications: We assume that (1) the designer is limited to a simple class of mechanisms that only allows price controls and lump-sum transfers (in a way we formalize soon), and (2) the agents’ rates of substitution are uniformly distributed. We show in Section 4 that the simple mechanisms we focus on are in fact optimal among all mechanisms satisfying natural incentive-compatibility, individual-rationality, market-clearing, and budget-balance constraints. Moreover, all of the qualitative conclusions we draw in this section extend to general distributions as long as appropriate regularity conditions hold, as we show in Section 5.9

Throughout this section, we assume that \( \lambda_j(r) \) is continuous and decreasing. The assumption that \( \lambda_j(r) \) is decreasing is of fundamental importance to our analysis: it captures the idea discussed earlier that the designer associates higher willingness to pay with lower expected value for money.10

### 3.1 Measures of inequality

We begin by introducing two measures of inequality that are central to our analysis. For \( j \in \{B, S\} \), we define

\[
\Lambda_j \equiv \mathbb{E}^j[v^M]
\]

to be buyers’ and sellers’ average values for money.

**Definition 1.** We say that there is cross-side inequality if buyers’ and sellers’ average values for money differ, i.e. if \( \Lambda_S \neq \Lambda_B \).

**Definition 2.** We say that there is same-side inequality on side \( j \in \{B, S\} \) (or just side-\( j \) inequality) if \( \lambda_j \) is not identically equal to \( \Lambda_j \). Same-side inequality is low for side \( j \) if \( \lambda_j(r_j) \leq 2\Lambda_j \); same-side inequality is high for side \( j \) if \( \lambda_j(r_j) > 2\Lambda_j \).

Cross-side inequality allows us to capture the possibility that agents on one side of the market are on average poorer than agents on the other side of the market. Meanwhile, same-side inequality captures the dispersion in wealth/value for money within each side of the market. To see this, consider the sellers: Under the assumption that \( \lambda_S(r) \) is decreasing, a seller with the lowest rate of substitution \( r_S \) is the poorest seller that can be identified based on her behavior in the marketplace—that is, she has the highest conditional expected value for money. Seller-side inequality is low if the poorest-identifiable seller has a conditional

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9Working with uniform distributions for now simplifies the analysis and allows us to deliver particularly sharp results.

10This assumption is fairly natural: Generating an increasing \( \lambda_j(r) \) would require a very strong positive correlation between \( v^K \) and \( v^M \).
expected value for money that does not exceed the average value for money by more than a factor of 2. The opposite case of high seller-side inequality implies that the poorest-identifiable seller has a conditional expected value for money that exceeds the average by more than a factor of 2. (The fact that that the threshold of 2 delineates qualitatively different solutions to the optimal design problem seems surprising; but in fact this threshold has a natural interpretation, as we explain in Section 3.6.)

3.2 Decomposition of the design problem

In our model, the only interaction between the buyer and seller sides of the market is due to the fact that (a) the market has to clear, and (b) the designer must maintain budget balance. Fixing both the quantity traded $Q$ and the revenue $R$, our problem decomposes into two one-sided design problems. To highlight key intuitions, we thus solve the design problem in three steps:

1. **Optimality on the seller side** – We identify the optimal mechanism that acquires $Q$ objects from sellers while spending at most $R$ (for any $Q$ and $R$).

2. **Optimality on the buyer side** – We identify the optimal mechanism that allocates $Q$ objects to buyers while raising at least $R$ in revenue (again, for any $Q$ and $R$).

3. **Cross-side optimality** – We identify the optimal market-clearing mechanism by linking our characterizations of seller- and buyer-side solutions through the optimal choice of $Q$ and $R$.

The proofs of the results in this section are omitted; in Appendix B.9, we show how these results follow as special cases of the more general results we establish in Sections 4 and 5.

3.3 Single-price mechanisms

At first, we allow the designer to choose only a single price $p_j$ for each side of the market.\(^\text{11}\) A given price determines supply and demand—and if there is excess supply or demand, then prospective traders are rationed uniformly at random until the market clears (reflecting the designer’s inability to observe the traders’ values directly). Moreover, the price has to be chosen in such a way that the designer need not subsidize the mechanism; if there is monetary surplus, that surplus is redistributed as a lump-sum transfer.

\(^{11}\)Here and hereafter, when we refer to a “price,” we mean a payment conditional on selling or obtaining the good, net of any lump-sum payment or transfer.
One familiar example of a single-price mechanism is the competitive mechanism, which, for a fixed quantity \( Q \), is defined by setting the price \( p_C^j \) that clears the market:

\[
G_S(p_C^S) = Q \quad \text{or} \quad \mu(1 - G_B(p_B^C)) = Q
\]

for sellers and buyers, respectively. Here, the word “competitive” refers to the fact that the ex-post allocation is determined entirely by agent’s choices based on their individual rates of substitution. In contrast, a “rationing” mechanism allocates the object with interior probability to some agents, with the ex-post allocation determined partially by randomization.

In a two-sided market, the competitive-equilibrium mechanism is defined by a single price \( p_{CE} \) that clears both sides of the market at the same (equilibrium) quantity:

\[
G_S(p_{CE}) = \mu(1 - G_B(p_{CE})).
\]

The competitive-equilibrium mechanism is always feasible; moreover, it is optimal when \( \lambda_j(r) = 1 \) for all \( r \), i.e., when the designer does not have redistributive preferences on both sides of the market.\(^{12}\)

**Optimality on the seller side**

We first solve the seller-side problem, determining the designer’s optimal mechanism for acquiring \( Q \) objects while spending at most \( R \). We assume that \( QG_S^{-1}(Q) \leq R \), as otherwise there is no feasible mechanism.

We note first that the designer cannot post a price below \( G_S^{-1}(Q) \) as there would not be enough sellers willing to sell to achieve the quantity target \( Q \). However, the designer can post a higher price and ration with probability \( Q/G_S(p_S) \). Thus, any seller willing to sell at \( p_S \), that is, with \( r \leq p_S \), gains utility \( p_S - r \) (normalized to units of money) with probability \( Q/G_S(p_S) \). Because each unit of money given to a seller with rate of substitution \( r \) is worth \( \lambda_S(r) \) in terms of social welfare, the net contribution of such a seller to welfare is \( (Q/G_S(p_S))\lambda_S(r)(p_S - r) \). Finally, with a price \( p_S \), buying \( Q \) units costs \( p_SQ \); if this cost is strictly less than \( R \), then the surplus can be redistributed as a lump-sum payment to all sellers. Since all sellers share lump-sum transfers equally, the marginal social surplus contribution of each unit of money allocated through lump-sum transfers is equal to the average value for money on the seller side, \( \Lambda_S \). Summarizing, the designer solves

\[
\max_{p_S \geq G_S^{-1}(Q)} \left\{ \frac{Q}{G_S(p_S)} \int_{p_S}^{p_S} \lambda_S(r)(p_S - r)dG_S(r) + \Lambda_S(R - p_SQ) \right\}.
\]  

\(^{12}\)As explained in Section 1.1, this follows from the first welfare theorem.
Uniform rationing has three direct consequences for social welfare: (i) allocative efficiency is reduced; (ii) the mechanism uses up more money to purchase the objects from sellers, leaving a smaller amount, $R - p_S Q$, to be redistributed as a lump-sum transfer; and (iii) those sellers who trade in the end receive a higher price. From the perspective of welfare, the first two effects are negative and the third one is positive; the following result describes the optimal resolution of this tradeoff.

**Proposition 1.** When seller-side inequality is low, it is optimal to choose $p_S = p_S^*$ (the competitive mechanism is optimal). When seller-side inequality is high, there exists an increasing function $\bar{Q}(R) \in [0, 1)$ (strictly positive for high enough $R$) such that rationing at a price $p_S > p_S^*$ is optimal if and only if $Q \in (0, \bar{Q}(R))$. Setting $p_S = p_S^*$ (i.e., using the competitive mechanism) is optimal otherwise.

Proposition 1 shows that when the designer is constrained to use a single price, competitive pricing is optimal (on the seller side) whenever seller-side inequality is low; meanwhile, under high seller-side inequality, rationing at a price above market-clearing becomes optimal when the quantity to be acquired is sufficiently low. As we show in Section 4, the simple mechanism described in Proposition 1 is in fact optimal among all incentive-compatible, individually-rational, budget-balanced, market-clearing mechanisms.

The key intuition behind Proposition 1 is that the decision to trade always identifies sellers with low rates of substitution: at any given price, sellers with low rates of substitution are weakly more willing to trade. By our assumption that $\lambda_S(r)$ is decreasing, we know that sellers with low rates of substitution are the poorest sellers that can be identified based on market behavior. Consequently, the trade-off between the effects (ii)—reducing lump-sum transfers—and (iii)—giving more money to sellers who trade—described before Proposition 1 is always resolved in favor of effect (iii): By taking a dollar from the average seller, the designer decreases surplus by the average value for money $\Lambda_S$, while giving a dollar to a seller who wants to sell at price $p$ increases surplus by

$$\mathbb{E}^S[\lambda_S(r) \mid r \leq p] \geq \Lambda_S$$

in expectation. However, to justify rationing, the net redistributive effect has to be stronger than the negative effect (i) on allocative inefficiency. When same-side inequality is low, the conditional value for money $\mathbb{E}^S[\lambda_S(r) \mid r \leq p]$ is not much higher than the average value $\Lambda_S$, even at low prices, so the net redistributive effect is weak. Thus, the negative effect of (i) dominates, meaning that the competitive price is optimal. When same-side inequality is high, however, the redistributive effect of rationing can dominate the effect of allocative inefficiency; Proposition 1 states that this happens precisely when the volume of trade is
sufficiently low. The intuition for why the optimal mechanism depends on the quantity of goods acquired is straightforward: When the volume of trade is low, only the sellers with the lowest rates of substitution sell—therefore, the market selection is highly effective at targeting the transfers to the agents who are most likely to be poor. In contrast, when the volume of trade is high, the decision to trade is relatively uninformative of sellers’ conditional values for money, weakening the net redistributive effect.

The threshold value of the volume of trade $\bar{Q}(R)$ depends on the revenue target $R$—when the budget constraint is binding, there is an additional force pushing towards the competitive price because that price minimizes the cost of acquiring the target quantity $Q$. It is easy to show that $\bar{Q}(R) > 0$ for any $R$ that leads to strictly positive lump-sum transfers (i.e., when the budget constraint is slack). At the same time, it is never optimal to ration when $Q$ approaches 1, because if nearly all sellers sell, then the market does not identify which sellers are poorer in expectation.

Optimality on the buyer side

We now turn to the buyer-side problem, normalizing $\mu = 1$ for this subsection as $\mu$ plays no role in the buyer-side optimality analysis. We assume that $QG_B^{-1}(1 - Q) \geq R$, as otherwise, there is no mechanism that allocates $Q$ objects to buyers while raising at least $R$ in revenue.

Similarly to our analysis on the seller side, we see that the designer cannot post a price above $G_B^{-1}(1 - Q)$, as otherwise there would not be enough buyers willing to purchase to achieve the quantity target $Q$. However, the designer can post a lower price and ration with probability $Q/(1 - G_B(p_B))$. Moreover, if the mechanism generates revenue strictly above $R$, the surplus can be redistributed to all buyers as a lump-sum transfer. Thus, analogously to (3.2), the designer solves

$$\max_{p_B \leq G_B^{-1}(1-Q)} \left\{ \frac{Q}{1 - G_B(p_B)} \int_{p_B}^{r_B} \lambda_B(r)(r - p_B)dG_B(r) + \Lambda_B(p_BQ - R) \right\}.$$  

Uniform rationing by setting $p_B < p^*_B$ has three direct consequences: (i) allocative efficiency is reduced, (ii) the mechanism raises less revenue, resulting in a smaller amount $(p_BQ - R)$ being redistributed as a lump-sum transfer, and (iii) the buyers that end up purchasing the good each pay a lower price. Like with the seller side of the market, the first two effects are negative and the third one is positive. Yet the optimal trade-off is resolved differently, as our next result shows.

**Proposition 2.** Regardless of buyer-side inequality, it is optimal to set $p_B = p^*_B$—that is, the competitive mechanism is optimal.
Proposition 2 shows that it is never optimal to ration the buyers at a single price below the market-clearing level—standing in sharp contrast to Proposition 1, which showed that rationing the sellers at a price above market-clearing is sometimes optimal.

The economic forces behind Propositions 1 and 2 highlight a fundamental asymmetry between buyers and sellers with respect to the redistributive power of the market: Whereas willingness to sell at any given price identifies sellers that have low rates of substitution and hence are poor in expectation, the buyers who buy at any given price are those that have higher rates of substitution and are hence relatively rich in expectation (recall that $\lambda_j(r)$ is decreasing). Effects (ii) and (iii) on the buyer side thus result in taking a dollar from an average buyer with value for money $\Lambda_B$ and giving it to a buyer (in the form of a price discount) with a conditional value for money

$$E^B[v_M \mid r \geq p] \leq \Lambda_B.$$

Thus, even ignoring the allocative inefficiency channel (i), under a single price the net redistribution channel decreases surplus.

### 3.4 Two-price mechanisms

We now extend the analysis of Section 3.3 by allowing the designer to introduce a second price on each side of the market. The idea is that the designer may offer a “market” price at which trade is guaranteed,\(^{13}\) and a “non-market” price that is more attractive (higher for sellers; lower for buyers) but induces rationing. Individuals self-select by choosing one of the options, enabling the designer to screen the types of the agents more finely than with a single price. For example, on the buyer side, the lowest-$r$ buyers will not trade, medium-$r$ buyers will select the rationing option, and highest-$r$ buyers will prefer to trade for sure at the “market” price.

**Optimality on the seller side**

As we noted in the discussion of Proposition 1, the simple single-price mechanism for sellers is in fact optimal among all feasible mechanisms. Thus, the designer cannot benefit from introducing a second price for sellers—at least under our the uniform distribution assumption we have made in this section.\(^{14}\)

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\(^{13}\)Here, we use “market” price informally to refer to a price that guarantees a purchase for an individual, not a price that clears the market.

\(^{14}\)In Section 4, we extend the results to a general setting and show that a second price may be optimal on the seller side for some distributions—nevertheless, the intuitions and conditions for optimality of rationing remain the same.
Optimality on the buyer side

As we saw in discussing Proposition 2, at any single price, the buyers who trade have a lower expected value for money than the buyer population average $\Lambda_B$; hence, lowering a (single) price redistributes money to a subset of buyers with lower contribution to social welfare. However, if the designer introduces a second price, she can potentially screen the buyers more finely. Suppose that the buyers can choose to trade at $p_{LB}$ with probability 1 or at $p_{LB}$ with probability $\delta < 1$ (thus being rationed with probability $1 - \delta$). Then, buyers with willingness to pay above $p_{LB}$ but below $r_\delta \equiv (p_{HB} - \delta p_{LB})/(1 - \delta)$ choose the rationing option while buyers with the highest willingness to pay (above $r_\delta$) choose the “market-price” option. Volume of trade is

$$1 - \delta G_B(p_{LB}) - (1 - \delta)G_B(r_\delta)$$

and revenue is

$$p_{LB}\delta(G_B(r_\delta) - G_B(p_{LB})) + p_{HB}(1 - G_B(r_\delta)).$$

Thus, to compute the optimal $p_{HB}^*, p_{LB}^*$, and $\delta$, the designer solves

$$\max_{p_{HB} \geq p_{LB}, \delta} \left\{ \delta \int_{r_\delta}^{r_B} \lambda_B(r)(r - p_{LB})dG_B(r) + \int_{r_\delta}^{p_{HB}} \lambda_B(r)(r - p_{HB})dG_B(r) + \Lambda_B(p_{LB}\delta(G_B(r_\delta) - G_B(p_{LB})) + p_{HB}(1 - G_B(r_\delta)) - R) \right\}$$

subject to the market-clearing and revenue-target constraints

$$1 - \delta G_B(p_{LB}) - (1 - \delta)G_B(r_\delta) = Q$$
$$p_{LB}\delta(G_B(r_\delta) - G_B(p_{LB})) + p_{HB}(1 - G_B(r_\delta)) \geq R.$$

We say that there is rationing at the lower price $p_{LB}$ if $\delta < 1$ and $G_B(r_\delta) > G_B(p_{LB})$, i.e., if a non-zero measure of buyers choose the lottery. With this richer class of mechanisms, we obtain the following result.

**Proposition 3.** When buyer-side inequality is low, it is optimal not to offer the low price $p_{LB}$ and to choose $p_{HB} = p_{LB}$ (the competitive mechanism is optimal). When buyer-side inequality is high, there exists a decreasing function $Q(R) \in (0, 1]$, strictly below 1 for low enough $R$, such that rationing at the low price is optimal if and only if $Q \in (Q(R), 1)$. Setting $p_{HB}^* = p_{LB}$ (and not offering the low price $p_{LB}$) is optimal for $Q \leq Q(R)$.

We show in Section 4 that the mechanism described in Proposition 3 is in fact optimal among all incentive-compatible, individually-rational, budget-balanced, market-clearing mechanisms.
The result of Proposition 3 relies on the fact that the decision to choose the rationing option identifies buyers that are poor in expectation. However, rationing only identifies poor-in-expectation buyers if inequality is substantial and sufficiently many (rich-in-expectation) buyers choose the high price; a large volume of trade ensures this because it implies that most buyers choose to buy for sure. In such cases, our mechanism optimally redistributes by giving a price discount to the buyers who value money the most.

The revenue target $R$ influences the threshold volume of trade $Q(R)$ above which rationing becomes optimal: If the designer needs to raise a lot of revenue, then rationing becomes less attractive. The threshold $Q(R)$ is strictly below 1 whenever the optimal mechanism gives a strictly positive lump-sum transfer. Even so, $Q(R)$ is never equal to 0—when almost no one buys, those who do buy must be relatively rich in expectation, and as a result rationing would (suboptimally) redistribute to wealthier buyers.

### 3.5 Cross-side optimality

Having found the optimal mechanisms for buyers and seller separately under fixed $Q$ and $R$, we now derive the optimal mechanism with $Q$ and $R$ determined endogenously.

**Proposition 4.** When same-side inequality is low on both sides of the market, it is optimal to set $p_B \geq p_S$ such that the market clears, $G_S(p_S) = \mu(1 - G_B(p_B))$, and redistribute the resulting revenue as a lump-sum payment to the side of the market $j \in \{B, S\}$ with higher average value for money $\Lambda_j$.

When same-side inequality is low, rationing on either side is suboptimal for any volume of trade and any revenue target (Propositions 1 and 3); hence, rationing is also suboptimal in the two-sided market. However, in order to address cross-side inequality, the mechanism may introduce a tax-like wedge between the buyer and seller prices in order to raise revenue that can be redistributed to the poorer side of the market. Intuitively, the size of the wedge (and hence the size of the lump-sum transfer) depends on the degree of cross-side inequality. For example, when there is no same-side inequality, and $\Lambda_S \geq \Lambda_B$, prices satisfy

$$p_B - p_S = \left(\frac{\Lambda_S - \Lambda_B}{\Lambda_S}\right) \frac{1 - G_B(p_B)}{g_B(p_B)}.$$  \hspace{1cm} (3.3)

Now, we suppose instead that there is high seller-side inequality. We know from Proposition 1 that rationing the sellers becomes optimal when the volume of trade is low. A sufficient condition for low volume of trade is that there are few buyers relative to sellers; in this case, rationing the sellers becomes optimal in the two-sided market.
**Proposition 5.** When seller-side inequality is high and $\Lambda_S \geq \Lambda_B$, if $\mu$ is low enough, then it is optimal to ration the sellers by setting a single price above the competitive-equilibrium level.

The assumption $\Lambda_S \geq \Lambda_B$ is needed in Proposition 5: If we instead had buyers poorer than sellers on average, the optimal mechanism might prioritize giving a lump-sum payment to buyers over redistributing among sellers. In that case, the optimal mechanism would minimize expenditures on the seller side—and as posting a competitive price is the least expensive way to acquire a given quantity $Q$, rationing would then be suboptimal.

As we saw in Proposition 3, rationing the buyers in the one-sided problem can be optimal if the designer introduces both a high price at which buyers can buy for sure and a discounted price at which buyers are rationed—however, for rationing to be optimal, we also require a high volume of trade. As it turns out, there are two-sided markets in which the optimal volume of trade is always relatively low, so that buyer rationing is suboptimal even under severe imbalance between the sizes of the two sides of the market—in contrast to Proposition 5.

**Proposition 6.** If seller-side inequality is low and $r_B = 0$, then the optimal mechanism does not ration the buyers.

To understand Proposition 6, recall that when we ration the buyer side optimally, we provide the good to relatively poor buyers at a discounted price. With $L_B = 0$ and high volume of trade (which is required for rationing to be optimal, by Proposition 3), revenue from the buyer side must be low. As a result, under rationing, buyers with low willingness to pay $r$ (equivalently, with high expected value for money) are more likely to receive the good, but at the same time they receive little or no lump-sum transfer. Yet, money is far more valuable than the good for buyers with $r$ close to $L_B = 0$. Thus, it is better to raise the price and limit the volume of trade—and hence increase revenue, thereby increasing the lump-sum transfer.

We assume low seller-side inequality in Proposition 6 in order to ensure that seller-side inequality does not make the designer want to raise the volume of trade. Under low seller-side inequality, the seller-side surplus is in fact decreasing in trade volume; thus, the designer chooses a volume of trade that is lower than would be chosen if only buyer welfare were taken into account.

The reasoning just described is still valid when $L_B$ is above 0 but not too large. However, rationing the buyers in the two-sided market may be optimal when all buyers’ willingness to pay is high, as formalized in the following result.
Proposition 7. If there is high buyer-side inequality, $\Lambda_B \geq \Lambda_S$, and
\[ \ell_B - \bar{r}_S \geq \frac{1}{2}(\bar{r}_B - \ell_S), \tag{3.4} \]
then there is some $\epsilon > 0$ such that it is optimal to ration the buyers for any $\mu \in (1, 1 + \epsilon)$.

The condition (3.4) in Proposition 7 is restrictive: It requires that the lower bound on buyers’ willingness to pay is high relative to sellers’ rates of substitution and relative to the highest willingness to pay on the buyer side. To understand the role of (3.4), recall that, by Proposition 3, a necessary and sufficient condition for buyer rationing in the presence of high buyer-side inequality is that a sufficiently high fraction of buyers trade. The condition (3.4) ensures that the optimal mechanism maximizes volume of trade because (i) there are large gains from trade between any buyer and any seller ($\ell_B$ is larger than $\bar{r}_S$), and (ii) it is suboptimal to limit supply to raise revenue ($\ell_B$ is large relative to $\bar{r}_B$). When $\mu \in (1, 1 + \epsilon)$ (there are slightly more potential buyers than sellers), maximal volume of trade means that almost all buyers buy, and hence rationing becomes optimal.

3.6 Why a factor of 2 in the definition of inequality?

We now offer intuition for why 2 is the threshold separating low and high same-side inequality—that is, why high same-side inequality obtains exactly when the trader with the lowest rate of substitution has a conditional value for money more than twice the average value. We focus on the seller side of the market, although an analogous intuition holds for the buyer side, as well.

With high seller-side inequality, Proposition 1 indicates that rationing is optimal at small volumes of trade (if the budget constraint is not too tight). To simplify notation, we assume that $r_S = 0$, and consider the welfare associated with posting a small price $p \approx 0$. As $p$ is small, we can treat $\lambda_S(r)$ as being approximately constant—equal to $\lambda_S(0)$—for $r \in [0, p]$.

If the budget constraint is not binding, then the opportunity cost of a unit of money spent on purchases of the object is the marginal value of the lump-sum transfer, $\Lambda_S$. Thus, the welfare gain from setting price $p$ is
\[ G_0 \equiv \int_0^p \lambda_S(r)(p - r)dG_S(r) \approx \lambda_S(0)g_S(0) \int_0^p (p - r)dr, \]
while the (opportunity) cost is
\[ C_0 \equiv \Lambda_S \cdot pG_S(p). \]

\[ \text{The budget constraint is slack in this case because the assumption } \ell_B > \bar{r}_S \text{ guarantees that any mechanism yields strictly positive revenue.} \]
Now, suppose that the designer considers introducing rationing by raising the price to \( p + \epsilon \) but keeping the quantity fixed, for some small \( \epsilon \). The gain is now

\[
G_1 \equiv \frac{G_S(p)}{G_S(p + \epsilon)} \int_0^{p+\epsilon} \lambda_S(r)[p + \epsilon - r]dG_S(r) \approx \lambda_S(0)g_S(0) \int_0^{p+\epsilon} (p + \epsilon - r) \frac{p}{p + \epsilon} dr,
\]

where \( \frac{G_S(p)}{G_S(p + \epsilon)} \) is the rationing coefficient, and the new opportunity cost is

\[
C_1 \equiv \Lambda_S \cdot (p + \epsilon)G_S(p) = C_0 + \epsilon \Lambda_S g_S(0)p.
\]

Rationing is optimal when the change in gains exceeds the change in costs:

\[
\Delta G \equiv \lambda_S(0) \frac{g(0)p}{\text{value for money}} \frac{1}{2} \epsilon \quad > \quad \Delta C \equiv \lambda_S \frac{g(0)p}{\text{value for money}} \frac{\epsilon}{\text{per agent cost}},
\]

that is, when \( \lambda_S(0) > 2\Lambda_S \). Intuitively, increasing the price received by sellers by \( \epsilon \) requires raising \( \epsilon \) in additional revenue. But when the designer increases price by \( \epsilon \), half of the resulting surplus is wasted because of inefficient rationing. Thus, for the switch to rationing to be socially optimal, it has to be that the agents who receive the extra \( \epsilon \) of money value it at least twice as much as do the agents who give it up.

This intuition is illustrated in Figure 3.1. The surplus \( G_0 \) associated with price \( p \) is given by the blue triangle ABC. The dotted red triangle AED illustrates the hypothetical surplus associated with raising the price to \( p + \epsilon \) without rationing—which increases surplus by an amount proportional to \( \epsilon \) (up to terms that are second-order in \( \epsilon \)). With rationing, the actual surplus is increased by an amount proportional to \( \frac{1}{2} \) and given by the area of the solid red triangle ABD (the seller with rate of substitution 0 is exactly indifferent between receiving a price \( p \) for sure and receiving the price \( p + \epsilon \) with probability \( \frac{p}{p+\epsilon} \) under rationing). The white area between the solid red triangle ABD and the dotted red triangle AED represents the surplus loss due to inefficient rationing. The figure depicts unweighted surplus—the actual contribution of the triangular areas to welfare is given by multiplying the area by the conditional value for money, which is approximately \( \lambda_S(0) \) when \( p \) is small. Rationing is optimal when

\[
\frac{\lambda_S(0) \cdot \epsilon}{2}
\]

exceeds the per-agent change in costs associated with the price increase from \( p \) to \( p + \epsilon \), which is

\[
\Lambda_S \cdot \epsilon.
\]

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Figure 3.1: The surplus (gross of lump-sum transfers) from posting a price $p$ (blue triangle ABC) versus from rationing at a price $p + \epsilon$ (red triangle ABD).

The intuition just presented illustrates, in particular, that the threshold of 2 does not depend on our uniform distribution assumption. Indeed, our reasoning only relied on local (first-order) changes, so all the calculations remain approximately valid for any distribution $G_S$ that has a positive continuous density around its lower bound $r_S$. For small changes in the price, the region of the surplus change is approximately a triangle, and hence the factor of 2 comes out of the formula for the area of a triangle.

4 Optimal Mechanisms – The General Case

In this section, we show how the insights we obtained in Section 3 extend to our general model. We demonstrate that even when the designer has access to arbitrary (and potentially complex) mechanisms, the optimal mechanism is quite simple, with only a few trading options available to market participants. Then, in Section 5, we show that our results about optimal market design under inequality continue to hold for general distributions of rates of substitution.

We assume that the designer can choose any trading mechanism subject only to four natural constraints: (1) Incentive-Compatibility (the designer does not observe individuals’ rates of substitution), (2) Individual-Rationality (each agent weakly prefers the outcome of
the mechanism to the status quo), (3) Market-Clearing (the volume of goods sold is equal to the volume of goods bought), and (4) Budget-Balance (the designer cannot subsidize the mechanism).

By the Revelation Principle, it is without loss of generality to look at direct mechanisms in which agents report their values and are incentivized to do so truthfully. This leads us to the following formal definition of a feasible mechanism.

**Definition 3.** A feasible mechanism \((X_B, X_S, T_B, T_S)\) consists of \(X_j : [v_j^K, \bar{v}_j^K] \times [\underline{v}_j^M, \bar{v}_j^M] \rightarrow [0, 1]\) and \(T_j : [v_j^K, \bar{v}_j^K] \times [\underline{v}_j^M, \bar{v}_j^M] \rightarrow \mathbb{R}\) for \(j \in \{B, S\}\), that satisfy the following conditions for all types \((v^K, v^M)\) and potential false reports \((\hat{v}^K, \hat{v}^M)\):

\[
X_B(v^K, v^M)v^K - T_B(v^K, v^M)v^M \geq X_B(\hat{v}^K, \hat{v}^M)v^K - T_B(\hat{v}^K, \hat{v}^M)v^M, \quad (IC-B)
\]

\[
-X_S(v^K, v^M)v^K + T_S(v^K, v^M)v^M \geq -X_S(\hat{v}^K, \hat{v}^M)v^K + T_S(\hat{v}^K, \hat{v}^M)v^M, \quad (IC-S)
\]

\[
X_B(v^K, v^M)v^K - T_B(v^K, v^M)v^M \geq 0, \quad (IR-B)
\]

\[
-X_S(v^K, v^M)v^K + T_S(v^K, v^M)v^M \geq 0, \quad (IR-S)
\]

\[
\int_{\underline{v}^K_B}^{\bar{v}^K_B} \int_{\underline{v}^M_B}^{\bar{v}^M_B} X_B(v^K, v^M)\mu dF_B(v^K, v^M) = \int_{\underline{v}^K_S}^{\bar{v}^K_S} \int_{\underline{v}^M_S}^{\bar{v}^M_S} X_S(v^K, v^M)dF_S(v^K, v^M), \quad (MC)
\]

\[
\int_{\underline{v}^K_B}^{\bar{v}^K_B} \int_{\underline{v}^M_B}^{\bar{v}^M_B} T_B(v^K, v^M)\mu dF_B(v^K, v^M) \geq \int_{\underline{v}^K_S}^{\bar{v}^K_S} \int_{\underline{v}^M_S}^{\bar{v}^M_S} T_S(v^K, v^M)dF_S(v^K, v^M). \quad (BB)
\]

We assume that the mechanism designer is utilitarian, and hence chooses an outcome to maximize welfare.

**Definition 4.** A mechanism \((X_B, X_S, T_B, T_S)\) is *optimal* if it is feasible and maximizes

\[
TV := \int_{\underline{v}^K_B}^{\bar{v}^K_B} \int_{\underline{v}^M_B}^{\bar{v}^M_B} [X_B(v^K, v^M)v^K - T_B(v^K, v^M)v^M] \mu dF_B(v^K, v^M)
\]

\[
+ \int_{\underline{v}^K_S}^{\bar{v}^K_S} \int_{\underline{v}^M_S}^{\bar{v}^M_S} [-X_S(v^K, v^M)v^K + T_S(v^K, v^M)v^M] dF_S(v^K, v^M) \quad (VAL)
\]

among all feasible mechanisms.

In our model, in general, direct mechanisms should allow agents to report their two-dimensional types, as in Definition 3. However, as we foreshadowed in Section 3, and as we formally show in Appendix A.1, it is without loss of generality to assume that agents only report their rates of substitution. Intuitively, reporting rates of substitution suffices because those rates fully describe individual agents’ preferences. (The mechanism could elicit information about both values by making agents indifferent between reports—but we
show that this can only happen for a measure-zero set of types and thus cannot raise the surplus achieved by the optimal mechanism.) Abusing notation slightly, we write \( X_j(v^K/v^M) \) for the probability that an agent with type \((v^K, v^M)\) trades object \(K\), and \( T_j(v^K/v^M) \) for the net change in the holdings of money. Moreover, again following the same reasoning as in Section 3, we can simplify the objective function of the designer to

\[
TV = \int_{\mathbb{L}_B} \lambda_B(r) [X_B(r)r - T_B(r)] \mu dG_B(r) + \int_{\mathbb{L}_S} \lambda_S(r) [-X_S(r)r + T_S(r)] dG_S(r),
\]

(VAL')

where \( G_j \) is the induced distribution of the rate of substitution \( r = v^K/v^M \), and \( \lambda_j(r) \) is the expectation of the value for money conditional on the rate of substitution \( r \),

\[
\lambda_j(r) = \mathbb{E}^j \left[ v^M | \frac{v^K}{v^M} = r \right].
\]

We assume that \( G_j(r) \) admits a density \( g_j(r) \) fully-supported on \([\underline{r}_j, \overline{r}_j] \).

### 4.1 Derivation of optimal mechanisms

We now present and prove our main technical result. While a mechanism in our setting can involve offering a menu of prices and quantities (i.e., transaction probabilities) for each rate of substitution \( r \), we nevertheless find that there is always an optimal mechanism with a relatively simple form.

To state our theorem, we introduce some terminology that relates properties of direct mechanisms to more intuitive properties of their indirect implementations: If an allocation rule \( X_j(r) \) takes the form \( X_B(r) = 1_{\{r \geq p\}} \) for buyers or \( X_S(r) = 1_{\{r \leq p\}} \) for sellers (for some \( p \)), then we call the corresponding mechanism a competitive mechanism, reflecting the idea that the (ex-post) allocation in the market depends only on agents’ behavior. An alternative to a competitive mechanism is a rationing mechanism which (at least sometimes) resorts to randomization to determine the final allocation: We say that side \( j \) of the market is rationed if \( X_j(r) \in (0, 1) \) for a non-zero measure set of types \( r \). Rationing for type \( r \) can always be implemented by setting a price \( p \) that is acceptable to \( r \) and then excluding \( r \) from trading with some probability. If \( n = |\text{Im}(X_j) \setminus \{0, 1\}| \), then we say that the mechanism offers \( n \) (distinct) rationing options to side \( j \) of the market; then,

\[
|\text{Im}(X_B) \setminus \{0, 1\}| + |\text{Im}(X_S) \setminus \{0, 1\}|
\]
is the total number of rationing options offered in the market. Finally, fixing \((X_B, X_S, T_B, T_S)\), we let \(U_j\) be the minimum utility among all types on side \(j\) on the market, expressed in units of money. Then, if \(U_j > 0\), we say that the mechanism subsidizes side \(j\)—the interpretation is that all agents on side \(j\) of the market receive a (positive) monetary lump-sum transfer of \(U_j\).

**Theorem 1.** Either:

- there exists an optimal mechanism that offers at most two rationing options in total and does not subsidize either side (i.e., \(U_S, U_B = 0\)), or
- there exists an optimal mechanism that offers at most one rationing option in total and that subsidizes the side of the market that has a higher average \(v^M\) (i.e., the side with a higher \(\Lambda_j\)).

Theorem 1 narrows down the set of candidate solutions to a class of mechanisms indexed by eight parameters: four prices, two rationing coefficients, and a pair of lump-sum payments. In particular, Theorem 1 implies that optimal redistribution can always be achieved by the use of lump-sum transfers and rationing. Moreover, if lump-sum redistribution is used, then rationing takes a particularly simple form: it is only used on one side of the market, and consists of offering a single rationing option. When lump-sum redistribution is not used, rationing could take a more complicated form, with either a single rationing option on each side of the market, or a competitive mechanism on one side, and two rationing options on the other side.

Except for the case in which two rationing options (and hence three prices) may be needed on one side of the market (which we can rule out with certain regularity conditions that we explore later), the simple two-price mechanism considered in Section 3 is sufficient to achieve the fully optimal market design under arbitrary forms of inequality.

### 4.2 Proof of Theorem 1

In this section, we explain the proof of Theorem 1, while relegating a number of details to the appendix.

---

16Here, \(\text{Im}(X_j)\) denotes the image of the function \(X_j\), and \(|A|\) denotes the cardinality of set \(A\).

17This interpretation is valid under the convention that, in any indirect implementation, prices belong to the range \([\underline{r}_j, \bar{r}_j]\) (which is always without loss of generality). For example, if buyers’ willingness to pay lies in \([1, 2]\) and the price in the market is 0 with all buyers trading, then \(U_B = 1\), we can equivalently set the price to 1 and think of buyers receiving a lump-sum transfer of 1 each.

18The mechanism is effectively characterized by five parameters, as lump-sum payments are pinned down by a binding budget-balance condition and the property that one of the lump-sum payments is 0. Also, the market-clearing condition for good \(K\) reduces the degrees of freedom on prices by 1.
Sketch

The key ideas of our argument are as follows. Any incentive-compatible individually-rational mechanism may be represented as a pair of lotteries over quantities (one for each side of the market). A lottery specifies the probability that a given type trades the good. Because of our large-market assumption, the lottery generates stochastic outcomes for individuals but deterministic outcomes in aggregate. With the lottery representation of the mechanism, the market-clearing condition (MC) states that the expected quantity must be the same under the buyer- and the seller-side lotteries. The objective function can thus be represented as expectation of a certain function of the realized quantity with respect to the pair of lotteries. If we furthermore incorporate the budget-balance constraint (BB) into the objective function by assigning a Lagrange multiplier, it follows that the value of the optimal lottery must be equal to the concave closure of the Lagrangian at the optimal trade volume; intuitively, for any fixed volume of trade, our problem becomes formally equivalent to a pair of “Bayesian persuasion” problems (one for each side of the market) with a binary state in which the market-clearing condition corresponds to the Bayes-plausibility constraint. Concavification then follows from the results of Aumann et al. (1995) and Kamenica and Gentzkow (2011). Looking separately at each side of the market, the optimal lottery over quantities concavifies a one-dimensional function (the Lagrangian) while satisfying a single linear constraint (the budget-balance condition). Hence, by Carathéodory’s Theorem, the optimal lottery for each side can be supported as a mixture over at most three points. To arrive at the conclusion of Theorem 1 that at most two rationing options are used in total, we exploit the fact that the two constraints (market-clearing and budget-balance) are common across the two sides of the market—looking at both sides of the market simultaneously allows us to further reduce the dimensionality of the solution by avoiding the double-counting of constraints implicit in solving for the buyers’ and sellers’ optimal lotteries separately.

Main argument

First, we simplify the problem by applying the canonical method developed by Myerson (1981), allowing us to express feasibility of the mechanism solely through the properties of the allocation rule and the transfer received by the worst type.\textsuperscript{19}

\textsuperscript{19}The proof of this step is skipped.
Claim 1. A mechanism \((X_B, X_S, T_B, T_S)\) is feasible if and only if

\[X_B(r)\text{ is non-decreasing in } r,\]  
\[X_S(r)\text{ is non-increasing in } r,\]  
\[\int_{r_B}^{r_B} X_B(r) \mu dG_B(r) = \int_{r_B}^{r_S} X_S(r) dG_S(r),\]  
\[\int_{r_B}^{r_B} J_B(r) X_B(r) \mu dG_B(r) - \mu U_B \geq \int_{r_S}^{r_S} J_S(r) X_S(r) dG_S(r) + U_S,\]

where \(J_B(r) \equiv r - \frac{1-G_B(r)}{g_B(r)}\) and \(J_S(r) \equiv r + \frac{G_S(r)}{g_S(r)}\) denote the virtual surplus functions, and \(U_B, U_S \geq 0\).

Second, using the preceding formulas and integrating by parts, we can show that the objective function \((\text{VAL}')\) also only depends on the allocation rule:

\[TV = \mu \Lambda_B U_B + \int \Pi_B^A(r) X_B(r) \mu dG_B(r) + \Lambda_S U_S + \int \Pi_S^A(r) X_S(r) dG_S(r),\]  
\[(\text{OBJ}')\]

where

\[\Pi_B^A(r) \equiv \frac{\int_{r}^{r_B} \lambda_B(r) dG_B(r)}{g_B(r)},\]  
\[\Pi_S^A(r) \equiv \frac{\int_{r}^{r_S} \lambda_S(r) dG_S(r)}{g_S(r)}.\]  

We refer to \(\Pi_j^A\) as the inequality-weighted information rents of side \(j\). In the special case of fully transferable utility, i.e., when \(\lambda_j(r) = 1\) for all \(r\), \(\Pi_j^A\) boils down to the usual information rent term, that is, \(G_S(r)/g_S(r)\) for sellers, and \((1-G_B(r))/g_B(r)\) for buyers.

Third, finding the optimal mechanism is hindered by the fact that the monotonicity constraints \((B\text{-Mon})\) and \((S\text{-Mon})\) may bind (“ironing” may be necessary, as shown by Myerson (1981)); in such cases, it is difficult to employ optimal control techniques. We get around the problem by representing allocation rules as mixtures over quantities; this allows us to optimize in the space of distributions and make use of the concavification approach.\(^{20}\) Because \(G_S\) has full support (it is strictly increasing), we can represent any non-increasing, right-continuous function \(X_S(r)\) as

\[X_S(r) = \int_0^1 1_{\{r \leq G_S^{-1}(q)\}} dH_S(q),\]

\(^{20}\)We thank Benjamin Brooks and Doron Ravid for teaching us this strategy.
where $H_S$ is a distribution on $[0, 1]$. Similarly, we can represent any non-decreasing, right-
continuous function $X_B(r)$ as

$$X_B(r) = \int_0^1 1_{(r \geq G_B^{-1}(1-q))} dH_B(q).$$

Economically, our representation means that we can express a feasible mechanism in the
quantile (i.e., quantity) space. To buy quantity $q$ from the sellers, the designer has to of-
fer a price of $G_S^{-1}(q)$, because then exactly sellers with $r \leq G_S^{-1}(q)$ sell. An appropriate
randomization over quantities (equivalently, prices) will replicate an arbitrary feasible quan-
tity schedule $X_S$. Similarly, to sell quantity $q$ to buyers, the designer has to offer a price
$G_B^{-1}(1-q)$, at which exactly buyers with $r \geq G_B^{-1}(1-q)$ buy. We have thus shown that it is
without loss of generality to optimize over $H_S$ and $H_B$ rather than $X_S$ and $X_B$ in (OBJ').

Fourth, we arrive at an equivalent formulation of the designer’s problem: Maximizing

$$\mu \int_0^1 \left( \int_{G_B^{-1}(1-q)}^{r_B} \Pi_B^A(r) dG_B(r) \right) dH_B(q) + \int_0^1 \left( \int_{\mathbb{X}_S}^{G_S^{-1}(1-q)} \Pi_S^A(r) dG_S(r) \right) dH_S(q) + \mu \Lambda_B U_B + \Lambda_S U_S$$

over $H_S, H_B \in \Delta([0, 1]), U_B, U_S \geq 0$, subject to

$$\mu \int_0^1 q dH_B(q) = \int_0^1 q dH_S(q),$$

$$\mu \int_0^1 \left( \int_{G_B^{-1}(1-q)}^{r_B} J_B(r) dG_B(r) \right) dH_B(q) - \mu U_B \geq \int_0^1 \left( \int_{\mathbb{X}_S}^{G_S^{-1}(1-q)} J_S(r) dG_S(r) \right) dH_S(q) + U_S.$$ 

Fifth, we can incorporate the constraint (4.5) into the objective function using a Lagrange
multiplier $\alpha \geq 0$. Defining

$$\phi_B^\alpha(q) \equiv \int_{G_B^{-1}(1-q)}^{r_B} (\Pi_B^A(r) + \alpha J_B(r)) dG_B(r) + (\Lambda_B - \alpha) U_B,$$

$$\phi_S^\alpha(q) \equiv \int_{\mathbb{X}_S}^{G_S^{-1}(1-q)} (\Pi_S^A(r) - \alpha J_S(r)) dG_S(r) + (\Lambda_S - \alpha) U_S,$$

our problem becomes one of maximizing the expectation of an additive function over two
distributions, subject to an inequality ordering the means of those distributions. We can

\footnote{Formally, considering all distributions $H_B$ and $H_S$ is equivalent to considering all feasible right-
continuous $X_B$ and $X_S$. The optimal schedules can be assumed right-continuous because a monotone function
can be made continuous from one side via a modification of a measure-0 set of points which thus does not
change the value of the objective function (OBJ').}
thus employ concavification to simplify the problem.

**Lemma 1.** Suppose that there exists $\alpha^* \geq 0$ and distributions $H_S^*$ and $H_B^*$ such that

$$
\int_0^1 \phi_S^\alpha(q)dH_S^*(q) + \mu \int_0^1 \phi_B^\alpha(q)dH_B^*(q) = \max_{Q \in [0, \mu \wedge 1], L_B, L_S \geq 0} \left\{ \text{co} \left( \phi_S^\alpha \right)(Q) + \mu \text{co} \left( \phi_B^\alpha \right)(Q/\mu) \right\},
$$

(4.6)

with constraints (4.4) and (4.5) holding with equality, where $\text{co}(\phi)$ denotes the concave closure of $\phi$, that is, the point-wise smallest concave function that lies above $\phi$. Then, $H_S^*$ and $H_B^*$ correspond to an optimal mechanism.

Conversely, if $H_B^*$ and $H_S^*$ are optimal, we can find $\alpha^*$ such that (4.4)–(4.6) hold.

Now, because the optimal $H_j^*$ found through Lemma 1 concavifies a one-dimensional function $\phi_j^\alpha$ while satisfying a linear constraint (4.5), Carathéodory’s Theorem implies that it is without loss of generality to assume that the lottery induced by $H_j^*$ has at most three realizations; this implies that the corresponding allocation rule $X_j$ has at most three jumps, and hence the mechanism offers at most two rationing options on each side of the market.

Finally, we can further reduce the dimensionality of the optimal solution by noticing that if two three-point lotteries $H_S$ and $H_B$ satisfy two constraints jointly—(4.4) and (4.5)—we can find an alternative pair of lotteries that also satisfy our constraints but put no mass on two out of the six points in their joint supports. This yields the characterization we state in the first part of Theorem 1. If additionally a strictly positive lump-sum payment is used for side $j$, then the Lagrange multiplier $\alpha^*$ must be equal to $\Lambda_j$; hence, the Lagrangian (4.6) must be constant in $U_j$. Starting from a mechanism offering two rationing options, we can find a mechanism with one rationing option such that (4.4) holds and the budget constraint (4.5) is satisfied as an inequality. Then, we can increase $U_j$ to satisfy (4.5) as an equality; the alternative solution still maximizes the Lagrangian, and is thus optimal, yielding the characterization we state in the second part of Theorem 1.

We note that Lemma 1 contains the key mathematical insight that allows us to relate the shape of $\phi_j^\alpha$ to the economic properties of the optimal mechanism: When $\phi_j^\alpha$ is concave, the optimal lottery is degenerate—corresponding to a competitive mechanism. When $\phi_j^\alpha$ is convex, it lies below its concave closure, and thus the optimal lottery is non-degenerate, leading to rationing.

The proof of Lemma 1, as well as the formal proofs of the preceding claims, are presented in Appendices B.1 and B.2, respectively.
5 Optimal Design under Inequality

In this section, we use the characterization of optimal mechanism derived in Section 4 to extend the conclusions of Section 3 to a large class of distributions satisfying certain regularity conditions.

5.1 Preliminaries

We maintain the key assumption that \( \lambda_j(r) \) is continuous and non-increasing in \( r \) but impose additional regularity conditions to simplify the characterization of our optimal mechanisms. First, we assume that the densities \( g_j \) of the distributions \( G_j \) of the rates of substitution are strictly positive and continuously differentiable (in particular continuous) on \([\underline{r}_j, \bar{r}_j] \), and that the virtual surplus functions \( J_B(r) \) and \( J_S(r) \) are non-decreasing. We make the latter assumption to highlight the role that inequality plays in determining whether the optimal mechanism makes use of rationing: With non-monotone virtual surplus functions, rationing (again, more commonly known in this context as “ironing”) can arise as a consequence of revenue-maximization motives implicitly present in our model due to the budget-balance constraint. We need an even stronger condition to rule out ironing due to irregular local behavior of the densities \( g_j \). To simplify notation, let \( \bar{\lambda}_j(r) = \lambda_j(r)/\Lambda_j \) for all \( r \) and \( j \) (we normalize so that \( \bar{\lambda}_j \) is equal to 1 in expectation), and define

\[
\Delta_S(p) \equiv \frac{\int_{\underline{r}_j}^{p} [\bar{\lambda}_S(\tau) - 1] g_S(\tau)d\tau}{g_S(p)} ,
\]

(5.1)

\[
\Delta_B(p) \equiv \frac{\int_{p}^{\bar{r}_j} [1 - \bar{\lambda}_B(\tau)] g_B(\tau)d\tau}{g_B(p)} .
\]

(5.2)

Assumption 1. The functions \( \Delta_S(p) - p \) and \( \Delta_B(p) - p \) are strictly quasi-concave in \( p \).

Unless otherwise specified, we impose Assumption 1 for the remainder of our analysis.

A sufficient condition for Assumption 1 to hold is that the functions \( \Delta_j(p) \) are (strictly) concave. Intuitively, concavity of \( \Delta_j(p) \) is closely related to non-increasingness of \( \lambda_j(r) \) (these two properties become equivalent when \( g_j \) is uniform, which is why Assumption 1 did not appear in Section 3). A non-increasing \( \lambda_j(r) \) reflects the belief of the market designer that agents with lower willingness to pay (lower \( r \)) are “poorer” on average, that is, have a higher conditional expected value for money. When \( \lambda_j(r) \) is assumed to be decreasing, concavity of \( \Delta_j \) rules out irregular local behavior of \( g_j \). Each function \( \Delta_j(p) \) is 0 at the endpoints \( \underline{r}_j \) and \( \bar{r}_j \), and non-negative in the interior. There is no same-side inequality if and only if \( \Delta_j(p) = 0 \) for all \( p \). We show in Appendix A.2 that, more generally, the functions \( \Delta_j \)
measure same-side inequality by quantifying the change in surplus associated with running a one-sided competitive mechanism with price \( p \) (which redistributes money from richer to poorer agents on the same side of the market).\(^{22}\)

### 5.2 Addressing cross-side inequality with lump-sum transfers

In this section, we show that lump-sum transfers are an optimal response of the market designer when cross-side inequality is significant, and that rationing is suboptimal when same-side inequality is low (recall the formal definitions in Section 3.1).

**Theorem 2.** Suppose that same-side inequality is low on both sides of the market. Then, the optimal mechanism is a competitive mechanism (with prices \( p_B \) and \( p_S \)).

A competitive-equilibrium mechanism, \( p_B = p_S = p^{CE} \), is optimal if and only if

\[
\Lambda_S \Delta_S(p^{CE}) - \Lambda_B \Delta_B(p^{CE}) \geq \begin{cases} 
(\Lambda_S - \Lambda_B) \frac{1 - G_B(p^{CE})}{g_B(p^{CE})} & \text{if } \Lambda_S \geq \Lambda_B \\
(\Lambda_B - \Lambda_S) \frac{G_S(p^{CE})}{g_S(p^{CE})} & \text{if } \Lambda_B \geq \Lambda_S.
\end{cases} \quad (5.3)
\]

When condition (5.3) fails, we have \( p_B > p_S \), and prices are determined by market-clearing \( \mu(1 - G_B(p_B)) = G_S(p_S) \), and, in the case of an interior solution,\(^{23}\)

\[
p_B - p_S = \begin{cases} 
-\frac{1}{\Lambda_S} \left[ \Lambda_S \Delta_S(p_S) - \Lambda_B \Delta_B(p_B) - (\Lambda_S - \Lambda_B) \frac{1 - G_B(p_B)}{g_B(p_B)} \right] & \text{if } \Lambda_S \geq \Lambda_B \\
-\frac{1}{\Lambda_B} \left[ \Lambda_S \Delta_S(p_S) - \Lambda_B \Delta_B(p_B) - (\Lambda_B - \Lambda_S) \frac{G_S(p_S)}{g_S(p_S)} \right] & \text{if } \Lambda_B \geq \Lambda_S.
\end{cases} \quad (5.4)
\]

The mechanism subsidizes the sellers (resp. buyers) when \( \Lambda_S > \Lambda_B \) (resp. \( \Lambda_B > \Lambda_S \)).

Theorem 2 is a generalization of Proposition 4 from Section 3. As we explained in Section 3, rationing is suboptimal when same-side inequality is low because the positive redistributive effects of rationing are too weak to overcome the allocative inefficiency that it induces. However, the optimal mechanism will often redistribute across the sides of the market if the difference in average values for money is sufficiently large; redistribution in this case takes the form of a tax-like wedge between the buyer and seller prices, which finances a lump-sum transfer to the poorer side of the market.\(^{24}\)

Condition (5.3), which separates optimality of competitive-equilibrium from optimality of lump-sum redistribution, depends on same-side inequality (through the term \( \Delta_j \)) because

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\(^{22}\)In Appendix A.2, we also give examples of distributions that satisfy our regularity conditions.

\(^{23}\)When no such interior solution exists, one of the prices is equal to the bound of the support: either \( p_B = \bar{r}_B \) or \( p_S = \bar{r}_S \).

\(^{24}\)If lump-sum transfers are not available, then rationing can sometimes arise as a second-best way of redistributing across the market (see Appendix A.3). We are presently undertaking a more general analysis of the case without lump-sum transfers in a follow-up paper, which also allows heterogeneous objects.
introducing a price wedge has redistributive consequences also within any side of the market. However, in the special case of no same-side inequality, Assumption 1 is automatically satisfied, (5.3) cannot hold unless $\Lambda_B = \Lambda_S$, and (5.4) boils down to (3.3), so that the wedge between the prices is proportional to the size of the cross-side inequality.

The proof of Theorem 2 relies on techniques developed in Section 4. A competitive mechanism corresponds to a one-step allocation rule which in turn corresponds to a degenerate lottery over quantities. A degenerate lottery is optimal exactly when the objective function—that is concavified in the optimal solution—is concave to begin with. Therefore, the key to the proof of Theorem 2 is to show that the Lagrangian $\phi_{\alpha_0}^*$ is concave under the assumption of low same-side inequality.

5.3 Addressing same-side inequality with rationing

A disadvantage of the competitive mechanism is that it is limited in how much wealth can be redistributed to the poorest agents. Indeed, market-clearing imposes bounds on equilibrium prices, and lump-sum transfers are shared equally by all agents on a given side of the market. When same-side inequality is low, a lump-sum transfer is a fairly effective redistributive channel. However, when same-side inequality is high, the conclusion of Theorem 2 may fail, as already demonstrated in Section 3. Here, we generalize these insights through a series of results. The first two results highlight and generalize the two main asymmetries between buyers and sellers identified in Section 3; the third result (extension of Proposition 5) gives sufficient conditions supporting seller-side rationing; the fourth and fifth result (extensions of Propositions 6 and 7) give sufficient conditions opposing and supporting buyer-side rationing, respectively.

**Theorem 3.** 1. For rationing to be optimal on the buyer side, the optimal volume of trade must be sufficiently large: $Q \geq Q_B > 0$ for some $Q_B$ that does not depend on the seller characteristics. Moreover, there must be a non-zero measure of buyers that trade with probability 1.

2. For rationing to be optimal on the seller side, the optimal volume of trade must be sufficiently small: $Q \leq Q_S < 1$ for some $Q_S$ that does not depend on the buyer characteristics. Moreover, there must be a non-zero measure of sellers that trade with probability 0.

Theorem 3 summarizes the asymmetry between buyers and sellers with respect to the redistributive properties of trading mechanisms. A competitive mechanism selects sellers who are poorest in expectation and buyers who are richest in expectation (due to our assumption
that $\lambda_j(r)$ is decreasing). Rationing on the seller side relies on identifying poor sellers directly via their decision to trade, and is successful only if relatively rich sellers are excluded from trading; this requires relatively low volume of trade. In contrast, to identify relatively poor buyers, a mechanism must offer two prices and attract sufficiently many rich buyers to the high “market” price (thus requiring a relatively large volume of trade). The intuitions just described are confirmed by the following result.

**Theorem 4.**
1. It is never optimal to ration buyers at a single price.\(^{25}\)
2. If either (i) the optimal mechanism subsidizes the sellers, or (ii) $J_S(G_S^{-1}(q))$ and $G_S^{-1}(q) - \Delta_S(G_S^{-1}(q))$ are convex in $q$, then rationing on the seller side (if optimal) takes the form of offering a single price above the market-clearing level.

Rationing at a single price is never optimal on the buyer side, as it would essentially amount to giving a price discount to the buyers that are relatively rich. In contrast, it is often optimal to ration the sellers at a single price. The assumptions required by the second part of Theorem 4 are restrictive because with general distributions it is difficult to predict how the optimal mechanism will be influenced by the budget-balance constraint (the Lagrange multiplier $\alpha$ is endogenous and influences the shape of the function $\phi_j^\alpha$); condition (i) addresses this difficulty by directly assuming that the optimal mechanism gives a lump-sum transfer to the sellers (this pins down a unique candidate for a Lagrange multiplier $\alpha$), while the alternative condition (ii) gives conditions on the primitives under which the form of the mechanism does not depend on how tight the budget constraint is (the key properties of $\phi_j^\alpha$ do not depend on the choice of $\alpha$). Condition (ii) is satisfied when $G_S$ is uniform.

Next, we turn attention to sufficient conditions for rationing to be optimal.

**Theorem 5.** Suppose that $\Lambda_S \geq \Lambda_B$ and seller-side inequality is high. Then, if $\mu$ is low enough (there are few buyers relative to sellers), the optimal mechanism rations the sellers.

Theorem 5 is a generalization of Proposition 5. When seller-side inequality is high, a low volume of trade is not only necessary but also sufficient for rationing to become optimal. Mathematically, this is because the function $\phi_S^\ast(q)$ is convex for low $q$ when seller-side inequality is high. Because the volume of trade is bounded above by the mass of buyers $\mu$, a low $\mu$ guarantees that at the optimal volume of trade $Q$, $\phi_S^\ast$ lies below its concave closure (see Lemma 1); this in turn implies that the optimal mechanism must correspond to a non-degenerate lottery $H_S^\ast$ over quantities (which is equivalent to rationing).

As we already explained in Section 3, there is no analog of Theorem 5 for buyers—it is possible that rationing is suboptimal regardless of the imbalance in the market. This happens

\(^{25}\)Formally, an optimal $X_B^\ast$ cannot satisfy $\text{Im}(X_B^\ast) \subseteq \{0, x\}$ for any $x < 1$. 

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in particular when the poorest buyers that can be identified by their market behavior have very low willingness to pay—in this case, providing the good at a “below-market” price (rationing) is always dominated by a competitive mechanism that redistributes money. We confirm this by extending Proposition 6.

**Theorem 6.** Suppose that same-side inequality for sellers is low, and \( \nu_B = 0 \). Then, if either (i) the optimal mechanism subsidizes the buyers, or (ii) \( G_B^{-1}(q) - \Delta_B(G_B^{-1}(q)) \) and \( J_B(G_B^{-1}(q)) \) are convex in \( q \), then the optimal mechanism does not ration the buyers.

Premise (i) holds when \( \bar{r}_S \geq \bar{r}_B \) and either (a) \( \Lambda_B \geq 2\Lambda_S \), or (b) there is no seller-side inequality and \( \Lambda_B \geq \Lambda_S \). Premise (ii) holds when \( G_B \) is uniform.

The economic intuition behind Theorem 6 is exactly as in Section 3. Here, we briefly explain the proof. When buyer-side inequality is high, the function \( \phi_B^{\alpha^*}(q) \) is convex for high enough \( q \), and thus rationing would be optimal for the buyer side if the volume of trade were large enough. However, under the assumption \( \nu_B = 0 \), we are able to show that \( \phi_B^{\alpha^*}(q) \) is decreasing whenever it is convex (the additional assumptions allow us to establish this property without necessarily knowing the value of the Lagrange multiplier \( \alpha^* \)). Therefore, from the perspective of buyer welfare, it is never optimal to choose a volume of trade in the region where rationing would become optimal (the maximum of buyer welfare is attained at a volume of trade lower than the one required for rationing). The assumption of low seller-side inequality implies that the optimal volume of trade overall is even lower than the optimal volume from the perspective of buyer welfare alone.

Theorem 6 crucially relies on existence of buyers with low willingness to pay. When all buyers value the good significantly, and it is relatively easy to ensure a large supply, assigning the good in a lottery at a “below-market” price may become an optimal policy. We demonstrate it be extending Proposition 7 to general distributions.

**Theorem 7.** Suppose that there is high buyer-side inequality and \( \Lambda_B \geq \Lambda_S \). There exists a constant \( M \) such that whenever \( \nu_B - \bar{r}_S \geq M \), it is optimal to ration the buyers for any \( \mu \in (1, 1 + \epsilon) \), for some \( \epsilon > 0 \).

The intuition behind Theorem 7 is the same as that for Proposition 7: The assumption of a large gap between buyer and seller values guarantees that it is optimal to sell all the goods supplied by sellers; with a slight size imbalance in the market, this means that almost all buyers buy, and hence rationing becomes optimal.

\[26\text{In particular, } M \leq \frac{1}{g_B(\bar{r}_B)} + \frac{1}{g_B(\bar{r}_S)}, \text{ which is finite by our assumption that the densities are strictly positive and continuous.}\]
6 Implications for Policy

As we noted in the Introduction, policymakers are actively engaged in redistributive market policies such as price controls that connect in many ways with the results described here. Economic analysis often opposes such regulations because they lead to allocative inefficiency. Yet our work suggests that such policies can in fact be part of the optimal design. Of course, our model is an extreme abstraction in many ways. Nevertheless, as we now describe, our basic assumptions are reasonable approximations of some real markets—and the framework provides at least intuitions for some others.

One real-world market that fits our model particularly well is the Iranian kidney market—the only cash market for kidneys in the world. In Iran, prospective kidney buyers and sellers register in a centralized market, mediated by the government. Humans have two kidneys, but need just one functional kidney to survive. Kidney buyers (i.e., end-stage renal disease patients) thus have unit demand for kidneys—which of course are indivisible goods. Prospective sellers quite literally have unit supply.\(^{27}\) The price of a kidney is fixed by the government, and as of July 2019, it is equivalent of 18 months of the minimum wage in Iran (see Ghods and Savaj (2006), Akbarpour et al. (2019)). Note that the pool of prospective buyers is completely separated from the pool of prospective sellers; moreover, essentially every individual who is not a buyer can potentially be a seller. Thus, seller-side inequality can be approximated by the inequality at the national level, which puts this market in the high seller-side inequality regime. Cross-side inequality, however, is relatively low, because the wealth distribution of the set of potential sellers (i.e., all citizens) is close to that of the buyers. In addition, since kidney patients are a tiny fraction of the population, the number of potential buyers is substantially less than the number of potential sellers. Therefore, Theorem 5 suggests that the policy of rationing the sellers at a single price corresponds to the optimal way to transfer surplus to the poorest prospective kidney sellers.

A second application that fits the outlines of our model is the rental real estate market; sellers in the rental market are the landlords, and the buyers are prospective renters. Of course, the rental real estate market has heterogeneous objects, which our model abstracts from. And changes in prices may induce income effects, which we also do not account for, since we assume agents have fixed marginal utilities for money. Nevertheless, our work provides some intuition for how we might think about addressing inequality in the rental housing market. In general, the sellers tend to be wealthier than the buyers (indeed, they own real estate equity). Moreover, there is tremendous buyer-side inequality. In addition,\(^{27}\)

\(^{27}\)While kidneys are not quite homogeneous due to blood-type differences, in Iran each blood-type submarket clears independently of the others; e.g., an AB blood-type buyer can only purchase kidneys from an AB blood-type seller.
the quantity of trade is high, since nearly all buyers need housing. Thus, the assumptions of Theorem 6 are unlikely to hold in this market. As Theorem 3 and Theorem 7 suggest, rationing buyers may be optimal: so long as most buyers rent at a high price, it might be optimal to provide a “lower-quality” rental option at a lower price—with the quality differential such that wealthier buyers do not want to mimic the poor buyers and buy at the low price. And indeed, some cities have faced problems when they make public housing too prevalent and high quality, since then even wealthy people claim public housing, reducing poorer people’s access. In such cases, rationing can be suboptimal, as it fails to identify poor buyers—see the reasoning behind our Proposition 3 and Theorem 3. Meanwhile, our model suggests that lump-sum transfers might be an effective strategy for addressing cross-side inequality in the rental market; such a policy could be implemented through a tax on rental transactions that is rebated as a tax credit to anyone who does not own housing (Diamond et al., 2019).

Our findings also provide some intuitions for labor markets. For example, our results suggest that rationing policies (e.g., local minimum wage policies) are most effective when there is high same-side inequality among potential workers, since transferring additional surplus to workers more than compensates for the allocative inefficiency. Thus, we might think than minimum wages make more sense for low-skilled jobs for which people from a wide variety of incomes can in principle participate. To address cross-side inequality (e.g., in the case of rideshare, where drivers are on average poorer than riders), lump-sum transfers are typically superior to rationing-based solutions. Of course, we should be cautious in extrapolating our results into labor market contexts because labor markets typically fail a number of our assumptions. Perhaps most pertinently, labor markets literally determine people’s incomes, yet our model rules out income effects by assuming fixed marginal utility for money. In addition, lump-sum transfers are difficult to implement in labor markets because the complete set of potential workers may be challenging to define (much less to target with transfer policies). Thus, for our results to provide more than just intuitions for labor markets, we need to think about closed, small economies such as work-study programs on university campuses. But of course even university labor markets have intensive and extensive labor supply decisions that put pressure on both the indivisibility assumption of our model and the assumption that the set of buyers and sellers is fixed ex ante.

28 There is of course also an extensive margin on the buyer side—prospective renters who might have the outside option of purchasing real estate.

29 This problem exists in, for instance, Amsterdam’s public housing system, where nearly 25% of households that live in public housing have incomes above the median (van Dijk, 2019).

30 That said, if these sorts of tax credits are available as an instrument, it is possible that broader, more efficient tax-based redistributive instruments may be available as well.
In general, Theorem 3, Theorem 6, and Theorem 7 suggest that rationing on the buyer side may only be justified for the “essential” goods such as housing and healthcare—goods that are highly valued by all potential buyers and that will induce a high quantity of trade. But of course—as with housing, already described—many essential goods are large enough to induce income effects, so we must be careful to interpret our results as just providing intuition for these markets, rather than a precise characterization of when rationing is optimal in practice. Additionally, our results on rationing rely on the indivisibility assumption; hence, they do not carry over directly to contexts like food aid, in which quantities are (almost) continuously divisible (although see our discussion of divisible goods in the next section). And meanwhile our results on lump-sum transfers are of course impracticable in markets in which lump-sum transfers are infeasible (as with open labor markets, already described); we discuss these issues briefly in Appendix A.3, and are undertaking a more general analysis of a setting with heterogeneous objects and without lump-sum transfers in a follow-up paper.

7 Discussion and Conclusion

Regulators often introduce price controls that distort markets’ allocative role in order to effect redistribution. Our work provides some justification for this approach, by showing that carefully structured price controls can indeed be an optimal response to inequality among market participants. The key observation, as we highlight here, is that properly designed price controls can identify poorer individuals through their behavior, using the marketplace itself as a redistributive tool.

Moreover, at least for a simple goods market, we can characterize the form that price controls should take. Our main result shows that optimal redistribution through markets can be obtained through a simple combination of lump-sum transfers and rationing. When there is substantial inequality between buyers and sellers, the optimal mechanism imposes a wedge between buyer and seller prices, passing on the resulting surplus to the poorer side of the market. When there is significant inequality on a given side of the market, meanwhile, the optimal mechanism may impose price controls even though doing so induces rationing. The form of rationing differs across the two sides: on the seller side, rationing at a single price can be optimal because willingness to sell identifies “poorer” sellers on average; by contrast, rationing the buyer side can only be optimal when several prices are offered, since buying at relatively high prices identifies agents that are likely to be wealthier.

Our paper does not examine how redistribution through markets interacts with macro-level redistribution. Theoretically, if macro-level redistribution already achieves the socially desired income distribution, then there is no scope for a market regulator with the same
preferences to try to improve the redistributive outcome by distorting the market allocation. Thus, we might naturally think that there is less need of market-level redistributive mechanisms in countries that either have large amounts of macro-level redistribution or simply have endogenously lower inequality—perhaps, that is, we should expect less need for market-based redistribution in a country like Sweden than in the United States. However, even in countries with low levels of inequality like Sweden, rent control is prevalent, perhaps more so than in the United States (see, e.g., Andersson and Söderberg (2012)). This could simply be because those societies have a higher preference for redistribution overall. Additionally, as our work highlights, market-based redistributive mechanisms can help screen for unobservable heterogeneity in the values for money that are not reflected in income. Finally, because our mechanisms are individually rational, they provide incentives for all agents to participate, whereas extremely progressive income taxation might raise extensive margin concerns, e.g., with wealthy individuals seeking tax havens. In any event, it remains an open question how much and when micro- and macro-level redistributive approaches are complements or substitutes.

The extent to which the mechanisms we derive here are valuable could also depend on policymakers’ preferences and political economy concerns: Market-level redistribution might be particularly natural in a context with divided government—either where the executive has a stronger preference for redistribution than the legislature, or where local policymakers have stronger preferences for redistribution than the central government.

The specific redistributive mechanisms we identify depend heavily on our assumptions—particularly, indivisibility of objects, unit demand, and linearity of agents’ utilities. We nevertheless expect that the core economic intuitions should carry over to settings with some of our assumptions relaxed. For example, in a model with a divisible good and utility that is concave in the quantity of the good but linear in money, we would expect rationing in the mechanism to be replaced with offering below-efficient quantities in order to screen for agents that are likely to be poorer. Likewise, if we were to instead relax quasi-linearity in money to allow for wealth effects (and hence endogenous Pareto weights that depend on the agents’ allocations and transfers), there would most likely still be opportunities to improve welfare through price wedges and/or rationing, but the optimal level of redistribution would be decreased because each unit of redistribution would also shift agents’ Pareto weights closer to equality.

That said, our framework abstracts from several practical considerations that are important in real-world settings. For instance, if there is an aftermarket (i.e., agents can engage in post- or outside-of-mechanism trades) then the mechanisms we consider might no longer be incentive-compatible (or budget-balanced). In addition, the generic form of our optimal
solution is a randomized mechanism, which can negatively affect the utilities of risk-averse agents and lead to wasted pre-market investments; both of these concerns are particularly salient in contexts with inequality, as the poor often have both less tolerance for day-to-day income variance and less ability to undertake upfront investments. Understanding how to redistribute through markets while accounting for these sorts of additional design constraints may be an interesting question for future research.

More broadly, there may be value in further reflecting on how underlying macroeconomic issues like inequality should inform market design. And we hope that the modeling approach applied here—allowing agents to have different marginal values of money—may prove useful for studying inequality in other microeconomic contexts.

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A Additional Discussions and Results

A.1 Equivalence between the two-dimensional value model and the Pareto weight model

In this appendix, we establish an equivalence between (i) our “two-dimensional” model, in which the designer maximizes total value (VAL) over feasible mechanisms according to Definition 3 and (ii) a “one-dimensional” model in which agents only report their rate of substitution \( r \) and the designer maximizes weighted surplus (with Pareto weights \( \lambda_j \)) according to (VAL’). While we only need one direction of the equivalence to justify the derivation of optimal mechanisms in Section 4, we demonstrate the full equivalence to show that we could just as well start our analysis with the one-dimensional model (with Pareto weights given as a primitive of the model) and our conclusions would remain identical.

To simplify notation, we use \((\bar{X}_B, \bar{X}_S, \bar{T}_B, \bar{T}_S)\) to denote a generic (direct) mechanism eliciting \((v^K, v^M)\) and \((X_B, X_S, T_B, T_S)\) to denote a generic (direct) mechanism eliciting \( r \). Formally, a mechanism \((X_B, X_S, T_B, T_S)\) is feasible in the one-dimensional model if for all \( r, \hat{r} \):

\[
X_B(r)r - T_B(r) \geq X_B(\hat{r})r - T_B(\hat{r}),
\]

\[
-X_S(r)r + T_S(r) \geq -X_S(\hat{r})r + T_S(\hat{r}),
\]

\[
X_B(r)r - T_B(r) \geq 0,
\]

\[
-X_S(r)r + T_S(r) \geq 0,
\]

\[
\int_{\xi_B}^{r_B} X_B(r)\mu dG_B(r) = \int_{\xi_S}^{r_S} X_S(r)dG_S(r),
\]

\[
\int_{\xi_B}^{r_B} T_B(r)\mu dG_B(r) \geq \int_{\xi_S}^{r_S} T_S(r)dG_S(r).
\]

A feasible mechanism \((X_B, X_S, T_B, T_S)\) is optimal in the one-dimensional model if it maximizes (VAL’) among all feasible mechanisms.

**Theorem 8.** If a mechanism \((X_B, X_S, T_B, T_S)\) is feasible (resp. optimal) in the two-dimensional model, then there exists a payoff-equivalent mechanism \((X_B, X_S, T_B, T_S)\) eliciting one-dimensional
reports that is feasible (resp. optimal) in the one dimensional model with $G_j$ equal to the distribution of $v^K/v^M$ under $F_j$ and $\lambda_j$ given by

$$\lambda_j(r) = \mathbb{E}_j \left[ v^M \mid \frac{v^K}{v^M} = r \right]. \quad (A.1)$$

Conversely, if a mechanism $(X_B, X_S, T_B, T_S)$ is feasible (resp. optimal) in the one-dimensional model, then there exists a joint distribution $F_j$ of $(v^K, v^M)$ such that this mechanism (with agents reporting $v^K/v^M$) is feasible (resp. optimal) in the two-dimensional model, $v^K/v^M$ has distribution $G_j$, and (A.1) holds.

**Proof.** We establish Theorem 8 in three steps:

1. We show that, without loss of generality, an incentive-compatible mechanism in the two-dimensional model only elicits information about the rate of substitution, $v^K/v^M$; thus, the space of feasible mechanisms is effectively the same in both settings.

2. We argue that the total value function (VAL) corresponds exactly to the objective function (VAL') with Pareto weights $\lambda_j(r)$ taken to be the expected value of $v^M$ conditional on observing a rate of substitution $r = v^K/v^M$.

3. As a consequence, if $G_j$ is the distribution of $v^K/v^M$ under $F_j$, and Pareto weights are defined as in Step 2, the same mechanism is optimal in both settings.

**Step 1.** We first formalize the idea that although agents have two-dimensional types, it is without loss of generality to consider mechanisms that only elicit information about the rate of substitution. While it is clear that the rate of substitution fully describes agents’ behavior, it could hypothetically be possible that the designer would elicit more information by offering different combinations of trade probabilities and transfers among which the agent is indifferent; we show, however, that this is only possible for a measure-zero set of agents’ types, and thus cannot strictly improve the designer’s objective.

**Lemma 2.** If $(X_B, X_S, T_B, T_S)$ is an incentive-compatible mechanism, then there exists a mechanism $(\bar{X}_B, \bar{X}_S, \bar{T}_B, \bar{T}_S)$ such that $\bar{X}_j(v^K, v^M) = X_j(v^K/v^M)$ and $\bar{T}_j(v^K, v^M) = T_j(v^K/v^M)$ for almost all $(v^K, v^M)$ and $j \in \{B, S\}$.

We prove Lemma 2 in Appendix A.1.1. Thanks to the lemma, and the assumption that there are no mass points in the distribution of values, we can assume (without loss of optimality) that agents report their rates of substitution $v^K/v^M$ in the two-dimensional model. Consequently, by direct inspection of the definition, the space of feasible mechanisms is the same in both models.
Suppose that the distribution $F_j$ and the weights $\lambda_j(r)$ are such that: $\Lambda_j = \mathbb{E}^j[v^M]$, for $j \in \{B, S\}$, and $\lambda_j(r)$ is given by (A.1). Moreover, let $G_j$ be the distribution of the random variable $v^K/v^M$ when $(v^K, v^M)$ is distributed according to $F_j$. Then, using Step 1, the objective functions (VAL) and (VAL') become identical:

$$
\mu \mathbb{E}^B \left[ X_B \left( \frac{v^K}{v^M} \right) v^K - T_B \left( \frac{v^K}{v^M} \right) v^M \right] + \mathbb{E}^S \left[ -X_S \left( \frac{v^K}{v^M} \right) v^K + T_S \left( \frac{v^K}{v^M} \right) v^M \right]
$$

$$
= \mu \mathbb{E}^B \left[ \mathbb{E}^{B, \lambda_B(r)} \left[ X_B (r) r - T_B (r) \right] \right] + \mathbb{E}^S \left[ \mathbb{E}^{S, \lambda_S(r)} \left[ -X_S (r) r + T_S (r) \right] \right].
$$

**Step 3.** The first part of Theorem 8 follows directly from preceding arguments. To prove the second part, we have to show that for any (fixed) $G_j$ and $\lambda_j(r)$, there exists a distribution $F_j$ of $(v^K, v^M)$ that induces that $G_j$ and $\lambda_j(r)$. The proof is simple: Fixing the random variable $r$ (on some probability space) with distribution $G_j(r)$, define random variables $v^K = r\lambda_j(r)$ and $v^M = \lambda_j(r)$. It is clear that the distribution of $v^K/v^M$ is the same as that of $r$ because these random variables are equal. Moreover, by construction, equation (A.1) must hold. □

**A.1.1 Proof of Lemma 2**

We start with the following result that provides a key step in the proof.

**Lemma 3.** Let $X(\theta_1, \theta_2)$ be a function defined on $[\theta_1, \bar{\theta}_1] \times [\theta_2, \bar{\theta}_2]$, with $\theta_1, \theta_2 \geq 0$, and assume that $X(\theta_1, \theta_2)$ is non-decreasing in $\theta_1/\theta_2$, that is

$$
\frac{\theta_1}{\theta_2} > \frac{\theta_1'}{\theta_2'} \implies X(\theta_1, \theta_2) \geq X(\theta_1', \theta_2').
$$

Then, there exists a non-decreasing function $x : \left[ \theta_1/\theta_2, \bar{\theta}_1/\bar{\theta}_2 \right] \to \mathbb{R}$ such that $X(\theta_1, \theta_2) = x(\theta_1/\theta_2)$ almost everywhere.

**Proof.** Consider $Y(r, \theta_2) = X(r\theta_2, \theta_2)$. For small enough $\epsilon > 0$ and almost all $r \in [\theta_1/\theta_2, \bar{\theta}_1/\bar{\theta}_2]$, $Y(r + \epsilon, \theta_2) \geq Y(r, \theta_2')$, $\forall \theta_2, \theta_2'$, by assumption. Because $Y(r, \theta_2)$ is non-decreasing in $r$ for every $\theta_2$, it is continuous in $r$ almost everywhere. Thus, for almost all $r$ we obtain

$$
Y(r, \theta_2) \geq Y(r, \theta_2'), \forall \theta_2, \theta_2'.
$$
this, however, means that \( Y(r, \theta_2) = x(r) \) for almost all \( r \) (does not depend on \( \theta_2 \)), for some function \( x \), that is moreover non-decreasing. Thus, \( X(r \theta_1, \theta_2) = x(r) \) for almost all \( r \). Therefore,

\[
X(\theta_1, \theta_2) = X \left( \frac{\theta_1}{\theta_2}, \theta_2 \right) = x \left( \frac{\theta_1}{\theta_2} \right)
\]

almost everywhere, as desired. \( \square \)

We now show that incentive-compatibility for buyers implies that \( \bar{X}_B(v^K, v^M) = X_B(v^K/v^M) \) for some non-decreasing \( X_B \). The argument for sellers is identical, and the statement for transfer rules follows immediately from the payoff equivalence theorem.

Incentive-compatibility means that for all \((v^K, v^M)\) and \((\hat{v^K}, \hat{v^M})\) in the support of \( F_B \) we have

\[
\tilde{X}_B(v^K, v^M) \frac{v^K}{v^M} - \bar{T}_B(v^K, v^M) \geq \tilde{X}_B(\hat{v^K}, \hat{v^M}) \frac{v^K}{v^M} - \bar{T}_B(\hat{v^K}, \hat{v^M}),
\]

as well as

\[
\tilde{X}_B(\hat{v^K}, \hat{v^M}) \frac{\hat{v^K}}{\hat{v^M}} - \bar{T}_B(\hat{v^K}, \hat{v^M}) \geq \tilde{X}_B(v^K, v^M) \frac{\hat{v^K}}{\hat{v^M}} - \bar{T}_B(v^K, v^M).
\]

(A.2)

(A.3)

Putting (A.2) and (A.3) together, we have

\[
(\tilde{X}_B(v^K, v^M) - \tilde{X}_B(\hat{v^K}, \hat{v^M})) \left( \frac{v^K}{v^M} - \frac{\hat{v^K}}{\hat{v^M}} \right) \geq 0.
\]

It follows that

\[
\frac{v^K}{v^M} > \frac{\hat{v^K}}{\hat{v^M}} \implies \tilde{X}_B(v^K, v^M) \geq \tilde{X}_B(\hat{v^K}, \hat{v^M}).
\]

By Lemma 3, it follows that there exists a non-decreasing \( X_B(\cdot) \) such that

\[
\tilde{X}_B(v^K, v^M) = X_B(v^K/v^M)
\]

almost everywhere, which finishes the proof.

### A.2 Interpreting our key regularity condition

We have imposed a relatively strong regularity condition (Assumption 1); in this section, we give an interpretation.

Observe that a competitive mechanism (in which we ignore market-clearing and only look at one side of the market) can be used to measure same-side inequality. Consider the seller side for concreteness, and suppose that we raise the price from \( p \) to \( p + \epsilon \). The following two
terms capture the associated gain in seller surplus:

\[ \epsilon \int_{r_S}^{p} \lambda_S(\tau) g_S(\tau) d\tau + \int_{p}^{p+\epsilon} \lambda_S(\tau)(p + \epsilon - \tau) d\tau. \]

That is, (1) sellers who were already selling at price \( p \) still sell at price \( p + \epsilon \), and hence receive an additional transfer of \( \epsilon \), which has expected value \( \lambda_S(\tau) \) for a seller with rate \( \tau \); and (2) sellers with \( \tau \in (p, p + \epsilon] \) decide to sell receiving the corresponding surplus. The social cost of increasing the price to \( p + \epsilon \) is that more revenue must be generated by the mechanism to cover the additional expenditure. If the shadow cost of revenue is \( \alpha \), then that cost is

\[ -\alpha [\epsilon G(p) + (p + \epsilon)(G(p + \epsilon) - G(p))], \]

where the expression in brackets is equal to the additional monetary transfer to sellers associated with the price increase. Because we are interested in measuring same-side inequality, the relevant shadow cost of revenue comes from charging sellers a lump-sum fee to cover the expenditures—thus, the shadow cost \( \alpha \) is equal to the average value for money \( \Lambda_S \) for sellers. Dividing by \( \epsilon \), and taking \( \epsilon \to 0 \), yields the local (first-order) net gain in seller surplus at price \( p \):

\[ \Lambda_S \left[ \int_{r_S}^{p} \frac{\tilde{\lambda}_S(\tau) g_S(\tau) d\tau}{g_S(p)} - \left( p + \frac{G_S(p)}{g_S(p)} \right) \right] g_S(p). \]  

(A.4)

The first bracketed term in (A.4) is typically increasing, especially for small \( p \) when \( \lambda_S(p) \) is relatively high, while the second bracketed term—minus the virtual surplus—is decreasing. Assumption 1 states that the bracketed expression in (A.4) can switch from being increasing to being decreasing at most once.\(^{31}\) Indeed, note that

\[ \int_{r_S}^{p} \frac{\tilde{\lambda}_S(\tau) g_S(\tau) d\tau}{g_S(p)} - \left( p + \frac{G_S(p)}{g_S(p)} \right) = \Delta_S(p) - p. \]

The function \( \Delta_S \) measures same-side inequality by quantifying the reduction in inequality associated with a competitive mechanism with price \( p \). The function \( \Delta_S \) is 0 at the extremes \( p \in \{r_S, \bar{r}_S\} \) (by definition of \( \tilde{\lambda}_S \)), and \( \Delta_S \) is non-negative because \( \lambda_S \) is decreasing. This is because a competitive mechanism with both the maximal (at which all sellers sell) and the minimal price (at which no sellers sell) does not reduce inequality. However, any interior competitive mechanism induces a monetary transfer to relatively poorer sellers. The area below the graph of \( \Delta_S \) (and above 0) can thus be seen as a compact measure of same-side inequality.

\(^{31}\) The reason why the density \( g_S(p) \) was pulled out from the expression in brackets is that in the full analysis we have to account for market-clearing, and thus the bracketed expression is weighted by the mass of sellers at a given price \( p \).
side inequality. In particular, $\Delta_S \equiv 0$ if and only if there is no same-side inequality. Our regularity assumption means that as the price moves between its two extreme values at which inequality is unchanged, the reduction in inequality changes in a monotone fashion.

Figure A.1 depicts the function $\Delta_S(p)$ derived under the assumption that $v^K$ is uniform on $[0, 1]$ and $v^M$ follows a Pareto distribution with tail parameter $\alpha > 2$, for different values of $\alpha$. The functions are concave, which is a sufficient condition for Assumption 1. The area below the curve is larger when $\alpha$ is smaller—the fatter the tail of agents with high value for money, the greater the inequality. It can be shown more generally that $\Delta_S(p)$ is concave when $v^K$ follows a distribution with CDF $F^K_S(v) = v^\beta$ for $\beta > 0$ and $\alpha > \beta$. Another sufficient condition for concavity of $\Delta_j(r)$, for $j \in \{B, S\}$, is that $g_B(p)$ is non-increasing, seller-side inequality is low, and $g_S(p)$ is non-decreasing.

### A.3 What if lump-sum redistribution is not feasible?

The optimal mechanism we identify often redistributes wealth through direct lump-sum transfers. Lump-sum transfers are natural in contexts in which the buyer and seller populations can be clearly defined according to characteristics that are either costly to acquire or completely exogenous—for example, if the only potential sellers are those who own land in a given area, or if the only eligible sellers are military veterans (as in some labor markets). Likewise, lump-sum transfers make sense when there is a licensing requirement or other rule that prevents agents from entering the market just to claim the transfer, or when the transfer can be made to an outside authority (e.g., a charity) that benefits the target population.

Nevertheless, lump-sum transfers may be difficult to implement in cases where buyer/seller participation is fully flexible. Indeed, imagine a mechanism that pays a constant amount to
sellers regardless of whether they trade. In such mechanisms, the sellers with the highest marginal rates of substitution might get strictly positive utility from participating in the marketplace even though they never trade; this creates an incentive for additional agents to enter the marketplace to reap the free benefit. Excess entry could then undermine the budget-balance condition. Consequently, we consider one additional constraint on the set of mechanisms.\footnote{One might think that a cleaner way to rule out the problem just described is to assume that a trader can only receive a transfer conditional on trading. Making transfers conditional on trading, however, would not work in our risk-neutral setting because an arbitrary expected transfer $T$ can be paid to an agent by paying her $T/\epsilon$ in the $\epsilon$-probability event of trading; because $\epsilon$ is arbitrarily small, this causes no distortion in the actual allocation.}

**Assumption 2 ("No Free Lunch").** The participation constraints of the lowest-utility type of buyers $r_B$, and of the lowest-utility type sellers $\bar{r}_S$ both bind, that is, types $r_B$ and $\bar{r}_S$ are indifferent between participating or not.

Under Assumption 2, the methods of the previous sections apply immediately, with the only modification that we set $U_S = U_B = 0$ in Lemma 1 (and hence $U_S$ and $U_B$ no longer appear as optimization parameters in the Lagrangian (4.6)). Theorem 1 then implies that there exists an optimal mechanism that offers at most two rationing options in total. The sufficient condition for optimality of competitive-equilibrium from Theorem 2 is still valid, since a competitive equilibrium mechanism does not use lump-sum transfers. However, the optimal mechanism from the second part of Theorem 2 is ruled out. Another novel aspect of the analysis is that we can no longer assume that the budget-balance condition binds at the optimal solution—Lemma 1 needs to be appropriately modified—because it is not feasible to distribute budget surplus using lump-sum transfers.

In general, the results of Section 5 need to be modified when Assumption 2 is imposed. When lump-sum transfers are ruled out, rationing is essentially the only tool available to the mechanism designer. In particular, rationing can sometimes be used to address cross-side inequality.

## B Omitted Proofs

In this section, we prove the results from Section 3–Section 5. Because the results in Section 3 are mostly corollaries of the general results derived in Section 5, we first prove the results of Section 4, then those of Section 5, and lastly those of Section 3.
B.1 Proof of Lemma 1

The constraint (4.5) satisfies the generalized Slater condition (see, e.g., Proposition 2.106 and Theorem 3.4 of Bonnans and Shapiro, 2000), so an approach based on putting a Lagrange multiplier $\alpha \geq 0$ on the constraint (4.5) is valid (strong duality holds). Moreover, we can assume without loss of generality that constraint (4.5) binds at the optimal solution (because $G_j$ admits a density, it follows that there exists a positive measure of buyers and sellers with strictly positive value for good $M$). This means that the problem (4.3)–(4.5) is equivalent to the following statement: There exists $\alpha^* \geq 0$ such that the solution to the problem

$$\max \left\{ \int_0^1 \phi^*_B(q)d(\mu H_B(q)) + \int_0^1 \phi^*_S(q)dH_S(q) \right\}$$

(B.1)

over $H_S, H_B \in \Delta([0, 1]), U_B, U_S \geq 0$, subject to

$$\int_0^1 qd(\mu H_B(q)) = \int_0^1 qdH_S(q)$$

(B.2)

satisfies constraint (4.5) with equality.

The value of the problem (B.1)–(B.2) can be computed by parameterizing

$$Q = \int_0^1 qd(\mu H_B(q))$$

and noticing that for a fixed $Q$, the choice of the optimal $H_S$ is formally equivalent to choosing a distribution of posterior beliefs in a Bayesian persuasion problem with two states, where equation (B.2) is the Bayes plausibility constraint. Hence, by Aumann et al. (1995) or Kamenica and Gentzkow (2011), the optimal distribution $H^*_S$ yields the value of the concave closure of $\phi^*_S$ at $Q$. Similarly, the optimal distribution $H^*_B$ yields the value of the concave closure of $\mu \phi^*_B$ at $Q/\mu$. Optimizing the value of the unconstrained problem

$$\text{co} \left( \phi^*_S \right)(Q) + \mu \text{co} \left( \phi^*_B \right)(Q/\mu)$$

over $Q, U_B, U_S \geq 0$ yields the optimal solution to the original problem if constraint (4.5) holds with equality at that solution. This gives the first part of the lemma.

Conversely, if $H^*_B$ and $H^*_S$ are optimal, then constraints (4.4)–(4.5) must bind. Optimality of $H^*_j$ implies that the value of $\phi^*_j$ at the optimum must be equal to its concave closure. We can define $Q = \int_0^1 qdH_S(q)$, and it must be that there exists $\alpha^* \geq 0$ such that $Q$ maximizes (B.1); this yields the second part of the lemma.
B.2 Completion of the proof of Theorem 1

First, we determine the optimal lump-sum transfers. Lemma 1 requires that the problem

$$\max_{Q \in [0,1], U_B, U_S \geq 0} \{ \text{co} \left( \phi_S^{\alpha^*} \right)(Q) + \mu \text{co} \left( \phi_B^{\alpha^*} \right)(Q/\mu) \}$$

has a solution, and this restricts the Lagrange multiplier to satisfy $\alpha^* \geq \max \{ \Lambda_S, \Lambda_B \}$. Indeed, in the opposite case, it would be optimal to set $U_j = \infty$ for some $j$ and this would clearly violate constraint (4.5). When $\Lambda_B = \Lambda_S$, it is either optimal to set $\alpha^* > \Lambda_S = \Lambda_B$ and satisfy (4.5) with equality and $U_S = U_B = 0$ (in which case there is no revenue and no redistribution), or to set $\alpha^* = \Lambda_S = \Lambda_B$ and $U_S = U_B \geq 0$ to satisfy (4.5) with equality (in which case the revenue is redistributed to both buyers and sellers as an equal lump-sum payment).\(^{33}\) When $\Lambda_B > \Lambda_S$, by similar reasoning, $U_S$ must be 0, and $U_B \geq 0$ is chosen to satisfy (4.5). When $\Lambda_S > \Lambda_B$, it is the seller side that obtains the lump-sum payment that balances the budget (4.5). In short, we can write the conditions for optimality of $U_S$ and $U_B$ as (ignoring the constraint (4.5) for now)

\begin{align}
U_S & \geq 0, \ U_S(\alpha^* - \Lambda_S) = 0; \quad (B.3) \\
U_B & \geq 0, \ U_B(\alpha^* - \Lambda_B) = 0. \quad (B.4)
\end{align}

Next, we consider the optimal lotteries $H_S^*$ and $H_B^*$. From Lemma 1, we know that each optimal lottery $H_j^*$ concavifies a one-dimensional function $\phi_S^\alpha$ while satisfying a single linear constraint (4.5). Therefore, by Carathéodory’s Theorem, we can assume without loss of generality that $H_j^*$ is supported on at most three points (an analogous mathematical observation in the context of persuasion was first made by Le Treust and Tomala, 2019, and is further generalized and explained in Doval and Skreta, 2018). We argue next that the dimension of the optimal pair of lotteries $(H_B^*, H_S^*)$ can be further reduced.

We denote $\psi_B(q) := \int_{G_B^1(1-q)} J_B(r) g_B(r) dr$ and $\psi_S(q) := \int_{L_S^1(q)} J_S(r) g_S(r) dr$. We then let $\text{supp}(H_j^*) = \{q_j^1, q_j^2, q_j^3\}$ with $q_j^1 \leq q_j^2 \leq q_j^3$. Observe that because the distribution $H_j^*$ concavifies $\phi_S^{\alpha^*}(q)$, it must be that $\text{co}(\phi_S^{\alpha^*})(q)$ is affine on the convex hull of the support of $H_j^*$. Moreover, because the volume of trade $Q \equiv \int_0^1 qdH_S^*(q)$ maximizes the concavified Lagrangian $\text{co} \left( \phi_S^{\alpha^*} \right)(q) + \mu \text{co} \left( \phi_B^{\alpha^*} \right)(q/\mu)$ over $q$, it follows that the Lagrangian is constant in $q$ on $\text{supp}(H_B^*) \cap \text{supp}(H_S^*)$, that is, on $[q, \bar{q}] \equiv [\max \{q_S^1, q_B^1\}, \min \{q_S^3, q_B^3\}]$. Thus, $Q \in [q, \bar{q}]$ and any volume of trade between $q$ and $\bar{q}$ is optimal.

The above reasoning, in particular the last observation, implies that the following linear

\(_{33}\)Of course, in this case, the surplus can also be redistributed only to the sellers, or only to the buyers, as long as condition (4.5) holds.
system admits a solution (here, the solution $\nu_j^i$ represents the realization probabilities of each $q_j^i$, while $U_S$ and $U_B$ satisfy (B.3) and (B.4)):

$$\mu \sum_{i=1}^3 \nu_B^i q_B^i = \sum_{i=1}^3 \nu_S^i q_S^i,$$

(B.5)

$$\mu \sum_{i=1}^3 \nu_B^i \psi_B(q_B^i) = \sum_{i=1}^3 \nu_S^i \psi_S(q_S^i) + \mu U_B + \mu U_S,$$

(B.6)

$$\sum_{i=1}^3 \nu_B^i = 1, \sum_{i=1}^3 \nu_S^i = 1,$$

(B.7)

$$\nu_B^1, \nu_B^2, \nu_B^3, \nu_S^1, \nu_S^2, \nu_S^3 \geq 0,$$

(B.8)

$$q \leq \sum_{i=1}^3 \nu_S^i q_S^i \leq \bar{q},$$

(B.9)

where (B.5) and (B.6) correspond to binding constraints (4.4) and (4.5), (B.7)–(B.8) state that we have well-defined probability measures, and (B.9) makes sure that the Lagrangian is maximized. From Lemma 1, any solution $(U_S, U_B, \nu)$ to the linear system (B.3)–(B.9) constitutes a solution to our original problem (by defining $H_j^*$ as putting a probability weight $\nu_j^i$ on $q_j^i$ for all $i$ and $j$). We can now establish the key claim.

**Claim 2.** Either:

- there exists a solution $(U_S, U_B, H_B^*, H_S^*)$ to (B.3)–(B.9) with $U_S = U_B = 0$ and

$$|\text{supp}(H_B^*)| + |\text{supp}(H_S^*)| \leq 4; \text{ or}$$

(B.10)

- there exists a solution $(U_S, U_B, H_B^*, H_S^*)$ to (B.3)–(B.9) with

$$|\text{supp}(H_B^*)| + |\text{supp}(H_S^*)| \leq 3.$$

(B.11)

**Proof.** We consider two cases. First suppose that $\alpha^* = \Lambda_j$ for some $j$. For concreteness and without loss of generality, we let $j = S$ and set $U_B = 0$; note that this automatically satisfies (B.4). Then, constraints (B.3) – (B.4) reduce to

$$U_S \geq 0.$$

(B.12)

The linear system (B.5)–(B.9), (B.12) has four equality constraints and seven free variables (six variable in the vector $\nu$ and $U_S$), and admits a solution. By the Fundamental Theorem of
Linear Algebra, there exists a solution in which seven constraints in the problem (B.5)–(B.9), (B.12) hold as equalities. Suppose first that (B.12) holds as an equality so that $U_S = 0$. Then, there exists a solution $(H^*_B, H^*_S)$ satisfying (B.10). Indeed, (B.10) is clear if the two additional binding constraints in the (sub)system (B.5)–(B.9) are constraints (B.8). In the alternative case when (B.9) binds, we conclude from $[q, \bar{q}] \equiv [\max\{q^1_S, q^1_B\}, \min\{q^3_S, q^3_B\}]$ that one of $H^*_j$ must be degenerate (supported on a singleton), so the claim follows as well.

Next, suppose that (B.12) holds as a strict inequality. Then, there exists a solution $(H^*_B, H^*_S)$ satisfying (B.11) because additional three inequalities must be equalities in the (sub)system (B.5)–(B.9).

Now consider the second case in which $\alpha^* > \Lambda_B, \Lambda_S$. Then, we must have $U_B = U_S = 0$ in all solutions. Thus, the linear system (B.5)–(B.9) has four equality constraints and six free variables (once $U_S$ and $U_B$ are fixed). By the same reasoning as above, there exists a solution $(H^*_B, H^*_S)$ satisfying (B.10). This finishes the proof of the claim.

Finally, we translate our results on the cardinality of the support of $(H^*_B, H^*_S)$ into our mechanism characterization.

**Claim 3.** If $|\text{supp}(H^*_B)| + |\text{supp}(H^*_S)| \leq m$, then the corresponding direct mechanism offers at most $m - 2$ rationing options in total.

**Proof.** We consider the seller side; the argument for the buyer side is analogous. Suppose that $|\text{supp}(H^*_S)| = n$. Let $r^k_S = G^{-1}_S(q^k_S)$, for all $k = 1, \ldots, n$. Then, the corresponding optimal $X_S(r)$ is given by

$$X_S(r) = \sum_{k=1}^{n} v^k_S 1_{\{r \leq r^k_S\}}.$$ 

By direct inspection, $|\text{Im}(X_S) \setminus \{0, 1\}| \leq n - 1$, so the conclusion follows.

Theorem 1 follows from Claim 3 and the restrictions on $|\text{supp}(H^*_B) \cup \text{supp}(H^*_S)|$ derived in Claim 2.

**B.3 Proof of Theorem 2**

We show that under the assumptions of the theorem, the functions $\phi^*_j$ are strictly concave with the optimal Lagrange multiplier $\alpha^*$. This is sufficient to prove optimality of a competitive mechanism because of Lemma 1—when the objective function is strictly concave, it coincides with its concave closure and the unique optimal distribution of quantities is degenerate, corresponding to a competitive mechanism.
As argued in the proof of Theorem 1, we must have \( \alpha^* \geq \max\{\Lambda_S, \Lambda_B\} \). Then, the derivative of the function \( \phi_S^*(q) \) takes the form

\[
(\phi_S^*)'(q) = \Pi_S^A(G_S^{-1}(q)) - \alpha^*J_S(G_S^{-1}(q)),
\]

so it is enough to prove that

\[
\Pi_S^A(r) - \alpha^*J_S(r)
\]

is strictly decreasing in \( r \). Rewriting (B.13), we have

\[
\Pi_S^A(r) - \alpha^*J_S(r) = \Lambda_S \left[ \frac{\int_{r_S}^r [\lambda_S(\tau) - 1]g_S(\tau)d\tau}{g_S(r)} - r \right] - (\alpha^* - \Lambda_S)J_S(r).
\]

Virtual surplus \( J_S(r) \) is non-decreasing, and \( \alpha^* \geq \Lambda_S \), so it is enough to prove that the first term is strictly increasing. The function \( \Delta_S(r) - r \) is strictly quasi-concave by Assumption 1, so to prove strict monotonicity on the entire domain, it is enough to show that the derivative at \( r = r_S \) is negative. We have

\[
\frac{d}{dr}[\Delta_S(r) - r]_{r=r_S} = \lambda_S(r_S) - 2 \leq 0,
\]

where the last inequality follows from the assumption that same-side inequality is low (recall that \( \Lambda_S\lambda_S(r_S) = \lambda_S(r_S) \)).

We now show that \( \phi_B^*(q) \) is also strictly concave:

\[
(\phi_B^*)'(q) = \Pi_B^A(G_B^{-1}(1 - q)) + \alpha^*J_B(G_B^{-1}(1 - q)),
\]

so it is enough to show that

\[
\Pi_B^A(r) + \alpha^*J_B(r)
\]

is increasing. Rewriting (B.14), we have

\[
\Pi_B^A(r) + \alpha^*J_B(r) = \Lambda_B [r - \Delta_B(r)] + (\alpha^* - \Lambda_B)J_B(r).
\]

Because the virtual surplus function \( J_B(r) \) is non-decreasing, and \( \alpha^* \geq \Lambda_B \), by assumption, it is enough to prove that \( r - \Delta_B(r) \) is increasing. Because this function is strictly quasi-convex by Assumption 1, it is enough to prove that the derivative is non-negative at the end point \( r = \bar{r}_B \):

\[
\frac{d}{dr}[r - \Delta_B(r)]_{r=\bar{r}_B} = 2 - \bar{\lambda}_B(\bar{r}_B) \geq 0,
\]
by the assumption that buyer-side inequality is low. Thus, we have proven that both functions \( \phi_j^{\alpha} \) are strictly concave.

It follows that a competitive mechanism with no rationing is optimal for both sides of the market, and the revenue (if strictly positive) is redistributed to the sellers if \( \Lambda_S \geq \Lambda_B \), and to the buyers otherwise (see Theorem 1). Concavity of \( \phi_j^{\alpha} \) implies that the first-order condition in problem (4.6) has to hold and is sufficient for optimality. This means that the optimal volume of trade \( Q^* \in [0, \mu \wedge 1] \) (the maximizer of the right hand side of (4.6)) satisfies

\[
\Pi^S_S(G_S^{-1}(Q^*)) - \alpha^* J_S(G_S^{-1}(Q^*)) + \Pi^B_B \left( G_B^{-1} \left( 1 - \frac{Q^*}{\mu} \right) \right) + \alpha^* J_B \left( G_B^{-1} \left( 1 - \frac{Q^*}{\mu} \right) \right) \geq 0
\]

(= 0 when \( Q^* < \mu \wedge 1 \)). (B.15)

Rewriting (B.15), and noting that \( p_S = G_S^{-1}(Q^*) \) and \( p_B = G_B^{-1}(1 - \frac{Q^*}{\mu}) \),

\[
\Lambda_S[\Delta_S(p_S) - p_S] - (\alpha^* - \Lambda_S) J_S(p_S) + \Lambda_B[p_B - \Delta_B(p_B)] + (\alpha^* - \Lambda_B) J_B(p_B) \geq 0
\]

(= 0 when \( Q^* < \mu \wedge 1 \)). (B.16)

Moreover, prices \( p_B, p_S \) have to satisfy \( p_B \geq p_S \) and clear the market:

\[
\mu(1 - G_B(p_B)) = G_S(p_S).
\]

First, assume that (5.3) holds at the competitive-equilibrium price \( p^{CE} \); we show that in this case, competitive-equilibrium is optimal. At \( p^{CE} \), market-clearing and budget-balance hold, by construction (with \( U_S = U_B = 0 \)). Therefore, we only have to prove existence of \( \alpha^* \geq \max\{\Lambda_S, \Lambda_B\} \) such that the first-order condition holds:

\[
\Lambda_S[\Delta_S(p^{CE}) - p^{CE}] - (\alpha^* - \Lambda_S) J_S(p^{CE}) + \Lambda_B[p^{CE} - \Delta_B(p^{CE})] + (\alpha^* - \Lambda_B) J_B(p^{CE}) \geq 0
\]

(B.18)

with equality when the solution is interior (i.e., when \( p^{CE} \in (\bar{r}_S, \bar{r}_B) \)). Simplifying (B.18) gives:

\[
\Lambda_S \left[ \Delta_S(p^{CE}) + \frac{G_S(p^{CE})}{g_S(p^{CE})} \right] - \Lambda_B \left[ \Delta_B(p^{CE}) - \frac{1 - G_B(p^{CE})}{g_B(p^{CE})} \right] \geq \alpha^* \left[ \frac{G_S(p^{CE})}{g_S(p^{CE})} + \frac{1 - G_B(p^{CE})}{g_B(p^{CE})} \right]
\]

with equality when \( p^{CE} \in (\bar{r}_S, \bar{r}_B) \). Since the left hand side is non-negative, such a solution \( \alpha^* \geq \max\{\Lambda_S, \Lambda_B\} \) exists if and only if we have an inequality at the minimal possible \( \alpha^* \),
that is, \( \alpha^* = \max\{\Lambda_S, \Lambda_B\} \):

\[
\Lambda_S \left[ \Delta_S(p^{CE}) + \frac{G_S(p^{CE})}{g_S(p^{CE})} \right] - \Lambda_B \left[ \Delta_B(p^{CE}) - \frac{1 - G_B(p^{CE})}{g_B(p^{CE})} \right] \\
\geq \max\{\Lambda_S, \Lambda_B\} \left[ \frac{G_S(p^{CE})}{g_S(p^{CE})} + \frac{1 - G_B(p^{CE})}{g_B(p^{CE})} \right].
\]

Simplifying the preceding expression shows that it is equivalent to condition (5.3).

It remains to show what the form the solution takes when condition (5.3) fails. A competitive equilibrium cannot be optimal in this case because there does not exist \( \alpha^* \) under which the corresponding quantity maximizes the Lagrangian (4.6) in Lemma 1. Consequently, we have \( p_B > p_S \), and, in light of Theorem 1, there will be a strictly positive lump-sum payment for the “poorer” side of the market: \( U_S > 0 \) when \( \Lambda_S \geq \Lambda_B \) and \( U_B > 0 \) when \( \Lambda_B > \Lambda_S \); this implies that we must have \( \alpha^* = \max\{\Lambda_S, \Lambda_B\} \). Subsequently, the optimal prices \( p_B \) and \( p_S \) are pinned down by market-clearing (B.17) and the first-order condition (B.16) which—assuming that an interior solution exists—becomes

\[
\Lambda_S(p_B - p_S) = -\Lambda_S \Delta_S(p_S) + \Lambda_B \Delta_B(p_B) + (\Lambda_S - \Lambda_B) \frac{1 - G_B(p_B)}{g_B(p_B)},
\]

when \( \Lambda_S \geq \Lambda_B \), and

\[
\Lambda_B(p_B - p_S) = -\Lambda_S \Delta_S(p_S) + \Lambda_B \Delta_B(p_B) + (\Lambda_B - \Lambda_S) \frac{G_S(p_S)}{g_S(p_S)},
\]

otherwise. When there is no interior solution, one of the prices is equal to the bound of the support of rates of substitution, and the other price is determined by the market-clearing condition. This finishes the proof of the theorem.

### B.4 Proof of Theorem 3

Consider the buyer side (we normalize \( \mu = 1 \) to simplify notation). We can decompose \( \phi_B^\alpha \) in the following way:

\[
\phi_B^\alpha(q) = \Lambda_B \int_{G_B^{-1}(1-q)}^{r_B} [r - \Delta_B(r)]g_B(r)dr + (\alpha^* - \Lambda_B) \int_{G_B^{-1}(1-q)}^{r_B} J_B(r)g_B(r)dr + (\Lambda_B - \alpha^*)U_B.
\]
Because the virtual surplus function is non-decreasing, the function \( \phi_2^B(q) \) is concave. Consider the function \( \phi_1^B \). We know that \( \phi_1^B(0) = 0 \), and that, by Assumption 1,

\[
(\phi_1^B)'(q) = G_B^{-1}(1 - q) - \Delta_B(G_B^{-1}(1 - q))
\]

is a quasi-convex function: It is non-increasing on \([0, \hat{q}]\) and non-decreasing on \([\hat{q}, 1]\), for some \( \hat{q} \in [0, 1] \). It follows that \( \phi_1^B(q) \) is concave on \([0, \hat{q}]\) and convex on \([\hat{q}, 1]\).

Consider \( \text{co}(\phi_1^B)(q) \). By the properties of \( \phi_1^B(q) \) described above, \( \text{co}(\phi_1^B)(q) \) is linear on an interval \([\tilde{q}, 1]\) for some \( \tilde{q} \), and \( \text{co}(\phi_1^B)(q) = \phi_1^B(q) \) for all \( q \leq \tilde{q} \). We show that \( \tilde{q} > 0 \). Suppose not, i.e., assume that \( \tilde{q} = 0 \), that is, the concave closure is a linear function supported at the endpoints of the domain. Then, since \( \phi_1^B \) is concave in the neighborhood of 0, it must be that a linear function tangent to \( \phi_1^B \) at \( q = 0 \) lies weakly below \( \phi_1^B \) at \( q = 1 \):

\[
\phi_1^B(0) + (\phi_1^B)'(0)(1 - 0) \leq \phi_1^B(1).
\]

Rewriting the above inequality, we obtain

\[
\bar{r}_B \leq \int_0^1 [G_B^{-1}(1 - q) - \Delta_B(G_B^{-1}(1 - q))]dq,
\]

or equivalently,

\[
\bar{r}_B \leq \int_{\underline{r}_B}^{\bar{r}_B} [r - \Delta_B(r)]dG_B(r),
\]

which is a contradiction since

\[
\int_{\underline{r}_B}^{\bar{r}_B} [r - \Delta_B(r)]dG_B(r) \leq \int_{\underline{r}_B}^{\bar{r}_B} rdG_B(r) < \bar{r}_B.
\]

The contradiction proves that \( \tilde{q} > 0 \). Finally, notice that since \( \alpha^* \geq \Lambda_B \) in the optimal mechanism, by Lemma 1, \( (\alpha^* - \Lambda_B)\phi_2^B(q) \) is a concave function which is added to \( \phi_1^B \) to obtain \( \phi_1^*_B \). Thus, the region in which \( \text{co}(\phi_1^*_B) \) is linear must be contained in the region where \( \text{co}(\phi_1^B) \) is linear (this follows directly from the definition of the concave closure). Therefore, \( \text{co}(\phi_1^*_B) \) cannot be linear on \([0, \hat{q}]\), and hence coincides with \( \phi_1^*_B(q) \) for \( q \in [0, \hat{q}] \).

We are ready to finish the first part of the proof. If there is rationing on the buyer side, then the optimal volume of trade must lie in the region where \( \phi_1^*_B \) lies strictly below its concave closure. It follows that \( Q \geq \hat{q} > 0 \), and that \( \text{supp}\{H_B^*\} \subseteq [\hat{q}, 1] \) (we can set \( Q_B = \hat{q} \)). This means that each corresponding price \( p_i = G_B^{-1}(1 - q^i) \) for \( q^i \in \text{supp}\{H_B^*\} \) satisfies \( p_i < \bar{r}_B \). Thus, there is non-zero measure of buyers who trade with probability one under the optimal mechanism.
Now consider the seller side. We can decompose \( \phi^*_S \) in a similar way:

\[
\phi^*_S(q) = \Lambda_S \int_{\mathcal{L}_S} [\Delta_S(r) - r] g_S(r) dr - (\alpha^* - \Lambda_S) \int_{\mathcal{L}_S} J_S(r) g_S(r) dr + (\Lambda_S - \alpha^*) U_S.
\]

Because the virtual surplus function is non-decreasing, the function \( \phi^1_S(q) \) is convex. Consider the function \( \phi^1_S \). We know that \( \phi^1_S(0) = 0 \), and that, by Assumption 1,

\[
(\phi^1_S)'(q) = \Delta_S(G_S^{-1}(q)) - G_S^{-1}(q)
\]

is a quasi-concave function: It is non-decreasing on \([0, \tilde{q}]\) and non-increasing on \([\tilde{q}, 1]\), for some \( \tilde{q} \in [0, 1] \). It follows that \( \phi^1_S(q) \) is convex on \([0, \tilde{q}]\) and concave on \([\tilde{q}, 1]\).

Consider \( \text{co}(\phi^1_S)(q) \). By the properties of \( \phi^1_S(q) \) described above, \( \text{co}(\phi^1_S)(q) \) is linear on some initial interval \([0, \tilde{q}]\) for some \( \tilde{q} \), and then \( \text{co}(\phi^1_S)(q) = \phi^1_S(q) \) for all \( q \geq \tilde{q} \). We show that \( \tilde{q} < 1 \). Suppose not, i.e., assume that \( \tilde{q} = 1 \), that is, the concave closure is a linear function supported at the endpoints of the domain. Then, since \( \phi^1_S \) is concave in the neighborhood of 1, it must be that a linear function tangent to \( \phi^1_S \) at \( q = 1 \) lies weakly below \( \phi^1_S \) at \( q = 0 \):

\[
\phi^1_S(1) + (\phi^1_S)'(1)(0 - 1) \leq \phi^1_S(0) = 0.
\]

Rewriting the above inequality, we obtain

\[
\int_0^1 [\Delta_S(G_S^{-1}(q)) - G_S^{-1}(q)] dq - [\Delta_S(G_S^{-1}(1)) - G_S^{-1}(1)] \leq 0
\]

or equivalently,

\[
\int_{\mathcal{L}_S} [\Delta_S(r) - r] dG_S(r) + \bar{r}_S \leq 0,
\]

which is a contradiction since

\[
\int_{\mathcal{L}_S} [\Delta_S(r) - r] dG_S(r) + \bar{r}_S \geq - \int_{\mathcal{L}_S} \bar{r} dG_S(r) + \bar{r}_S > 0.
\]

The contradiction proves that \( \tilde{q} < 1 \). Finally, notice that since \( \alpha^* \geq \Lambda_S \) in the optimal mechanism, by Lemma 1, \(- (\alpha^* - \Lambda_S) \phi^2_S(q) \) is a concave function which is added to \( \phi^1_S \) to obtain \( \phi^*_S \). Thus, the region in which \( \text{co}(\phi^*_S) \) is linear must be contained in the region where \( \text{co}(\phi^1_S) \) is linear. Therefore, \( \text{co}(\phi^*_S) \) cannot be linear on \([\tilde{q}, 1]\), and hence coincides with \( \phi^*_S(q) \) for \( q \in [\tilde{q}, 1] \).
We can now finish the second part of the proof. If there is rationing on the seller side, then the optimal volume of trade must lie in the region where $\phi^\star_{\alpha S}$ lies strictly below its concave closure. It follows that $Q \leq \tilde{q} < 1$, and that $\text{supp}\{H^\star_{\alpha S}\} \subseteq [0, \tilde{q}]$ (we can set $\tilde{Q}_S = \tilde{q}$). This means that each corresponding price $p^i = G^{-1}_S(q^i)$ for $q^i \in \text{supp}\{H^\star_{\alpha S}\}$ satisfies $p^i < \bar{r}_S$. Thus, there is non-zero measure of sellers who trade with probability zero under the optimal mechanism.

### B.5 Proof of Theorem 4

The first part of Theorem 4 follows from Theorem 3: If there is rationing on the buyer side, there must exist a non-zero measure of buyers that trade with probability 1—and thus it is never optimal to ration at a single price.

To prove the second part of the theorem, it is enough to prove that the function $\phi^\alpha_{\alpha S}(q)$ is first convex and then concave, for any $\alpha \geq \Lambda_S$. Indeed, this implies that the concave closure of $\phi^\alpha_{\alpha S}(q)$ is a linear function on $[0, \hat{q}]$ for some $\hat{q} > 0$, and coincides with $\phi^\alpha_{\alpha S}(q)$ otherwise. Thus, when there is rationing, it takes the form of a lottery between the quantities $q = 0$ and $q = \hat{q}$ which corresponds to a single price with rationing.

We prove that the derivative of $\phi^\alpha_{\alpha S}(q)$ is quasi-concave. Recall from the proof of Theorem 3 that

$$(\phi^\star_{\alpha S})'(q) = \Lambda_S(\phi^1_{\alpha S})'(q) - (\alpha^* - \Lambda_S)(\phi^2_{\alpha S})'(q).$$

Under assumption (i), sellers receive a strictly positive lump-sum transfer and hence we must have $\alpha^* = \Lambda_S$. At the same time we have

$$(\phi^1_{\alpha S})'(q) = \Delta_S(G^{-1}_S(q)) - G^{-1}_S(q)$$

which is quasi-concave by the regularity condition (a composition of a quasi-concave function with an increasing function is quasi-concave). Under assumption (ii), $(\phi^1_{\alpha S})'(q)$ is a concave function, and

$$(\phi^2_{\alpha S})'(q) = J_S(G^{-1}_S(q))$$

is a convex function. Thus, the derivative of $\phi^\alpha_{\alpha S}(q)$ is concave, and hence quasi-concave.

### B.6 Proof of Theorem 5

First, we prove a property of the function $\phi^\alpha_S$. Importantly, with $\alpha$ treated as a free parameter, $\phi^\alpha_S$ is determined by the primitive variables and does not depend on $\mu$. 


Lemma 4. There exist \( \hat{q} > 0 \) and \( \hat{\alpha} > \Lambda_S \) such that if \( \alpha < \hat{\alpha} \), then \( \phi^*_S(q) \) is strictly convex on \([0, \hat{q}]\).

Proof. The derivative of \( \phi^*_S(q) \) is \( \Pi^1_S(G^{-1}_S(q)) - \alpha J_S(G^{-1}_S(q)) \). Because the function \( G^{-1}_S(q) \) is strictly increasing, it is enough to prove that \( \Pi^1_S(r) - \alpha J_S(r) \) is strictly increasing for \( r \in [\underline{r}_S, \hat{r}] \), for some \( \hat{r} \) (we then set \( \hat{q} = G_S(\hat{r}) \)). Taking a derivative again, and rearranging, yields the following sufficient condition: for \( r \in [\underline{r}_S, \hat{r}] \),

\[
\lambda_S(r) > 2 + \frac{g_S(r)}{g_S(r)} \Delta_S(r) + \frac{\alpha - \Lambda_S}{\Lambda_S} \left[ 2 - \frac{g_S(r)G_S(r)}{\frac{g_S(r)}{g_S(r)}} \right].
\]

Because \( g_S \) was assumed continuously differentiable and strictly positive, including at \( r = \underline{r}_S \), we can put a uniform (across \( r \)) bound \( M < \infty \) on \( \frac{g_S(r)}{g_S(r)} \) and \( 2 - \frac{g_S(r)G_S(r)}{\frac{g_S(r)}{g_S(r)}} \). This means that it is enough that

\[
\lambda_S(r) > 2 + M \Delta_S(r) + \frac{\alpha - \Lambda_S}{\Lambda_S} M.
\]

Continuity of \( \lambda_S(r) \) and the assumption that seller-side inequality is high imply that \( \lambda_S(r) > 2 + \epsilon \) for \( r \in [\underline{r}_S, \underline{r}_S + \delta] \) for some \( \delta > 0 \). Continuity of \( \Delta_S(r) \) and the fact that \( \Delta_S(\underline{r}_S) = 0 \) imply that \( \Delta_S(r) < \epsilon/(3M) \) for all \( r \in [\underline{r}_S, \underline{r}_S + \nu] \) for some \( \nu > 0 \). Finally, there exists a \( \bar{\alpha} > \Lambda_S \) such that for all \( \alpha < \bar{\alpha} \), we have \( (\alpha - \Lambda_S)/\Lambda_S < \epsilon/(3M) \). Then, for all \( r \in [\underline{r}_S, \underline{r}_S + \min\{\delta, \nu\}] \), \( \alpha < \bar{\alpha} \),

\[
\lambda_S(r) > 2 + \epsilon > 2 + M \Delta_S(r) + \frac{\alpha - \Lambda_S}{\Lambda_S} M.
\]

The proof is finished by setting \( \hat{r} = \underline{r}_S + \min\{\delta, \nu\} \).

We now prove Theorem 5. Suppose that the optimal mechanism for sellers is a competitive mechanism. We derive a contradiction when \( \mu \) is low enough. There are two cases to consider: Either (1) \( \alpha^* = \Lambda_S \), or (2) \( \alpha^* > \Lambda_S \).

Consider case (1). We can invoke Lemma 4: Because \( \alpha^* = \Lambda_S \), there exists \( \hat{q} > 0 \) such that \( \phi^*_S(q) \) is strictly convex on \([0, \hat{q}]\). For small enough \( \mu \), namely \( \mu < \hat{q} \), we must have \( Q \leq \hat{q} \) since the volume of trade is bounded above by the mass of buyers. But then, at the optimal quantity \( Q \), \( \phi^*_S(Q) \) cannot be equal to its concave closure, and hence the optimal mechanism cannot be a competitive mechanism, contrary to our supposition. The obtained contradiction means that we must have case (2) when \( \mu \) is lower than \( \hat{q} \).

Consider case (2). Suppose that the optimal competitive mechanism for sellers has a price \( p^S \). By Theorem 1, we can assume that at most two rationing options are optimal on the buyer side; this corresponds to some prices \( p^B_1 \leq p^B_2 \leq p^B_3 \), and corresponding
quantities \( q_1^B \geq q_2^B \geq q_3^B \) that comprise the support of \( H_B^\mu \). Because the functions \( \text{co}(\phi^*_j) \) are differentiable, we know that the first-order condition of the problem (4.6) must hold at the optimal \( Q \):
\[
(\text{co}(\phi^*_S))^\prime(Q) + (\text{co}(\phi^*_B))^\prime(Q/\mu) \geq 0
\]
with equality for \( Q < \mu \). Moreover, from the definition of the concave closure, using the fact that \( Q \leq q_1^B \),
\[
(\text{co}(\phi^*_B))^\prime(Q/\mu) \leq (\phi^*_B)^\prime(q_1^B/\mu),
\]
with equality if \( q_1^B < 1 \). Similarly, on the seller side,
\[
(\text{co}(\phi^*_S))^\prime(Q) = (\phi^*_S)^\prime(Q)
\]
Therefore, we obtain
\[
(\phi^*_S)^\prime(Q) + (\phi^*_B)^\prime(q_1^B/\mu) \geq 0.
\]
Substituting \( G_S(p^S) = Q \) and \( \mu(1 - G_B(p_1^B)) = q_1^B \), we obtain
\[
\Lambda_S \left[ \Delta_S(p^S) + \frac{G_S(p^S)}{g_S(p^S)} \right] - \Lambda_B \left[ \Delta_B(p_1^B) - \frac{1 - G_B(p_1^B)}{g_B(p_1^B)} \right] \geq \alpha^* \left[ p^S - p_1^B + \frac{G_S(p^S)}{g_S(p^S)} + \frac{1 - G_B(p_1^B)}{g_B(p_1^B)} \right].
\]
Now, consider what happens as \( \mu \to 0 \). Since the market must clear, we must have \( p^S_\mu \to \underline{r}_S \) as \( \mu \to 0 \). Indeed, otherwise, there would be a positive (bounded away from zero) measure of sellers trading despite the fact that total volume of trade goes to zero. Therefore, writing the above expression in the limit as \( \mu \to 0 \), we obtain (using the fact that \( G_S(\underline{r}_S) = \Delta(\underline{r}_S) = 0 \),
\[
-\Lambda_B \left[ \Delta_B(p_1^B) - \frac{1 - G_B(p_1^B)}{g_B(p_1^B)} \right] \geq \alpha^* \left[ \underline{r}_S - p_1^B + \frac{1 - G_B(p_1^B)}{g_B(p_1^B)} \right],
\]
where, with slight abuse of notation, \( p_1^B \) denotes the limit as \( \mu \) goes to zero.\(^{35}\) By assumption, \( \alpha^* > \Lambda_S \) along the sequence, so we know that there cannot be any lump-sum transfers in the optimal mechanisms. Thus, budget-balance requires that the seller price \( \underline{r}_S \in [p_1^B, p_3^B] \). This implies that
\[
(\Lambda_B - \alpha^*) \frac{1 - G_B(p_1^B)}{g_B(p_1^B)} \geq \Lambda_B \Delta_B(p_1^B) \geq 0.
\]
Because \( \alpha^* > \Lambda_B \), and \( G_B(p_1^B) < 1 \) in the optimal mechanism, we obtain a contradiction.\(^{36}\)

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\(^{34}\)We use the fact that for small enough \( \mu \), \( Q \) must lie in the interior of \( [\underline{r}_S, \bar{r}_S] \).

\(^{35}\)We can assume that the limit exists because the domain of \( p_1^B \) is compact.

\(^{36}\)Formally, we have to exclude the possibility that \( p_1^B = \bar{r}_B \). There are two cases. If a competitive mechanism is optimal for buyers, then \( p_1^B = \underline{r}_S \) by budget-balance, and hence \( p_1^B < \bar{r}_B \) because the supports of buyer and seller rates overlap. If rationing is optimal on the buyer side, then prices are bounded away.
B.7 Proof of Theorem 6

By Theorem 2, we know that rationing cannot be optimal for buyers when there is low buyer-side inequality, so we can assume that buyer-side inequality is high without loss of generality.

Let \( \phi_B^1(q) \) and \( \phi_B^2(q) \) be defined as in proof of Theorem 3, and normalize \( \mu = 1 \) (it plays no role in this part of the proof). Recall that

\[
(\phi_B^1)'(q) = G_B^{-1}(1 - q) - \Delta_B(G_B^{-1}(1 - q)).
\]

The function \( r - \Delta_B(r) \) is strictly quasi-convex by Assumption 1. The function \( G_B^{-1}(1 - q) \) is strictly decreasing. A composition of strictly quasi-convex function with a strictly decreasing function is strictly quasi-convex. Therefore, \( (\phi_B^1)'(q) \) is strictly decreasing on \([0, \bar{q}]\) and strictly increasing on \([\bar{q}, 1]\) for some \( 0 \leq \bar{q} \leq 1 \). Moreover, \( (\phi_B^1)'(0) = \bar{r}_B > 0 \) and \( (\phi_B^1)'(\bar{q}) = \bar{r}_B = 0 \). It follows that \( (\phi_B^1)'(q) \) is negative whenever it is increasing, and thus \( \phi_B^1(q) \) is decreasing whenever it is convex. Because \( \phi_B^1(0) = 0 \) and \( (\phi_B^1)'(0) > 0 \), it follows that \( \phi_B^1(q) \) is (strictly) concave on \([0, q^*]\), where \( q^* \) achieves the global maximum of \( \text{co}(\phi_B^1)(q) \) over all \( q \in [0, 1] \).

We now prove that the above property of \( \phi_B^1(q) \) continues to hold for \( \phi_B^{α^*}(q) \).

**Lemma 5.** Suppose that the function \( \text{co}(\phi_B^{α^*}) \) has a global maximum at \( Q^* \). Then, \( \phi_B^{α^*} \) is strictly concave on \([0, Q^*]\) (and in particular equal to \( \text{co}(\phi_B^{α^*}) \)).

**Proof.** The proof differs depending on which assumption, \( (i) \) or \( (ii) \), is satisfied.

\( (i) \) When buyers receive a strictly positive lump-sum transfer, then we must have \( α^* = \Lambda_B \). It follows that \( \phi_B^{α^*}(q) = \Lambda_B \phi_B^1(q) + (\Lambda_B - α^*)U_B \), and hence \( \phi_B^{α^*} \) immediately inherits the required property from \( \phi_B^1 \).

\( (ii) \) We have \( (\phi_B^2)'(q) = J_B(G_B^{-1}(1 - q)) \), and thus \( (\phi_B^2)'(q) \) is convex by assumption. Similarly, \( (\phi_B^1)'(q) \) is convex by assumption. Therefore \( (\phi_B^{α^*})'(q) \) is convex, and moreover \( (\phi_B^{α^*})'(1) \leq 0 \). Therefore, \( \phi_B^{α^*} \) has the same property as \( \phi_B^1(q) \) (by the same argument). \( \square \)

The proof of the first part of the theorem now follows from Lemma 5. First, notice that \( \phi_S^α(q) \) (and hence also \( \text{co}(\phi_S^α)(q) \)) is non-increasing in \( q \) for any \( α \geq \Lambda_S \) under the assumptions of the theorem. This follows from \( (\phi_S^α)'(0) = -α\ell_S \leq 0 \) and the proof of Theorem 2 where we showed that under the regularity condition and low seller-side inequality, \( (\phi_S^α)'(q) \) is strictly decreasing. This implies that \( Q \), the maximizer of the Lagrangian (4.6), must be lower than \( Q^* \)—the maximizer of \( \text{co}(\phi_B^{α^*}) \) from Lemma 5. But then, by Lemma 5, \( \phi_B^{α^*}(q) \) is strictly decreasing whenever it is convex. Because \( \phi_B^{α^*}(0) = 0 \) and \( (\phi_B^{α^*})'(0) > 0 \), it follows that \( \phi_B^{α^*}(q) \) is (strictly) concave on \([0, q^*]\), where \( q^* \) achieves the global maximum of \( \text{co}(\phi_B^{α^*})(q) \) over all \( q \in [0, 1] \).
concave on \([0, Q^*]\) and coincides with its concave closure at \(q = Q\). Thus, there cannot be rationing on the buyer side.

We prove the second part of Theorem 6 by showing that the Lagrange multiplier can be taken to be \(\alpha^* = \Lambda_B\) in this case (we no longer assume that \(\mu = 1\) as this is not without loss for this part of the proof). From the first part of the proof, we know that with \(\alpha = \Lambda_B\), the function \(\phi_B^\alpha(q)\) is first concave and then convex, and that it achieves its global maximum on the part of the domain where it is concave. Because seller-side inequality is low (so that \(\phi_S^\alpha(q)\) is decreasing and concave), it is sufficient to prove that the first-order condition is satisfied (see the proof of Theorem 2),

\[
\Lambda_S[\Delta_S(p_S) - p_S] - (\Lambda_B - \Lambda_S)J_S(p_S) + \Lambda_B[p_B - \Delta_B(p_B)] = 0, \quad (B.19)
\]

the market clears,

\[
\mu(1 - G_B(p_B)) = G_S(p_S), \quad (B.20)
\]

and budget-balance is maintained: Because we aim to prove that a competitive mechanism is optimal for both sides, and \(\alpha^* = \Lambda_B\) implies that \(U_B\) can be an arbitrary positive number, it is enough if we prove that

\[
p_B \geq p_S. \quad (B.21)
\]

Thus, we seek to prove existence of a solution \((p_B^*, p_S^*)\) to the system \((B.19) - (B.20)\) which additionally satisfies \((B.21)\). First, notice that \((B.20)\) can be equivalently written as

\[
p_S = \psi(p_B) \equiv G_S^{-1}(\mu(1 - G_B(p_B))), \quad p_B \in [\underline{p}_B, \bar{r}_B],
\]

where \(\underline{p}_B = G_B^{-1}(\max(0, 1 - \frac{1}{\mu}))\) (when \(\mu > 1\), there cannot exist a solution in which \(p_B < \underline{p}_B\)). Therefore, we can write a single equation for \(p \in [\underline{p}_B, \bar{r}_B]\) as

\[
\Phi(p) \equiv \Lambda_S[\Delta_S(\psi(p)) - \psi(p)] - (\Lambda_B - \Lambda_S)J_S(\psi(p)) + \Lambda_B[p - \Delta_B(p)] = 0.
\]

The function \(\Psi(p)\) is continuous in \(p\), and we have

\[
\Phi(\bar{r}_B) = \Lambda_S[\Delta_S(r_S) - r_S] - (\Lambda_B - \Lambda_S)J_S(r_S) + \Lambda_B[\bar{r}_B - \Delta_B(\bar{r}_B)] = -\Lambda_B \underline{r}_S + \Lambda_B \bar{r}_B > 0.
\]

There are two cases to consider. When \(\mu \leq 1\), we have \(\underline{p}_B = \underline{L}_B, \quad \psi(\underline{p}_B) = G_S^{-1}(\mu)\), and thus

\[
\Phi(\underline{p}_B) \leq \Lambda_B \underline{L}_B = 0,
\]
by assumption. In the opposite case \( \mu > 1 \), we have \( p_B = G_B^{-1}(1 - \frac{1}{\mu}) \), \( \psi(p_B) = \bar{r}_S \), and thus

\[
\Phi(p_B) \leq -\Lambda_B \bar{r}_S + \Lambda_B G_B^{-1} \left( 1 - \frac{1}{\mu} \right) \leq -\Lambda_B \bar{r}_S + \Lambda_B \bar{r}_B \leq 0,
\]

using the assumption that \( \bar{r}_S \geq \bar{r}_B \). In both cases we conclude that \( \Phi(p_B) \leq 0 \). Because the function \( \Phi \) changes sign, there exists \( p_B^* \) such that \( \Phi(p_B^*) = 0 \), and then \( p_S^* = \psi(p_B^*) \) is well defined as well.

It remains to prove that this solution \( (p_B^*, p_S^*) \) satisfies (B.21). Rewrite the first-order condition as

\[
p_B - p_S = \Delta_B(p_B) - \frac{\Lambda_S}{\Lambda_B} \Delta_S(p_S) + \frac{\Lambda_B - \Lambda_S G_S(p_S)}{\Lambda_B g_S(p_S)}.
\]

Under assumption (b), there is no seller-side inequality and thus \( \Delta_S(p_S) \equiv 0 \). Because \( \Delta_B(p_B) > 0 \) and \( \Lambda_B \geq \Lambda_S \), we conclude that \( p_B > p_S \). Under assumption (a), we have

\[
p_B - p_S \geq \Delta_B(p_B) - \frac{1}{2} \Delta_S(p_S) + \frac{1}{2} G_S(p_S) \geq \frac{1}{2} J_B \left[ 2 - \bar{\lambda}_S(r) \right] dG_S(r) > 0,
\]

because seller-side inequality is low (\( \bar{\lambda}_S(r) \leq 2 \) for all \( r \)).

This finishes the proof: The fact that \( p_B > p_S \) implies that there is a strictly positive revenue in the mechanism, and the fact that \( \alpha^* = \Lambda_B \) implies that the revenue in the optimal mechanism is redistributed as a lump-sum payment to buyers.

### B.8 Proof of Theorem 7

Under the assumptions of Theorem 7, we have that \( \bar{r}_B > \bar{r}_S \); thus, any feasible (in particular optimal) mechanism must feature a strictly positive lump-sum transfer and \( \alpha^* = \Lambda_B \geq \Lambda_S \) (by the proof of Theorem 1). We prove that \( \mu \text{co}(\phi_B^*)(Q/\mu) + \text{co}(\phi_S^*)(Q) \) is increasing in \( Q \). Set \( M = 1/g_B(\bar{r}_B) + 1/g_S(\bar{r}_S) \)—a finite constant.

When \( \alpha^* = \Lambda_B \), we have

\[
(\phi_B^*)'(q) = \Lambda_B \left[ G_B^{-1}(1 - q) - \Delta_B(G_B^{-1}(1 - q)) \right] \geq \Lambda_B \left[ G_B^{-1}(1 - q) - \frac{1 - G_B(G_B^{-1}(1 - q))}{g_B(G_B^{-1}(1 - q))} \right]
= \Lambda_B J_B(G_B^{-1}(1 - q)) \geq \Lambda_B J_B(\bar{r}_B),
\]

where the last inequality follows from the fact that virtual surplus is monotone by assump-
tion. Hence, we have

$$\inf_q \left\{ \frac{d}{dq} [\mu \text{co}(\phi_B^*(q/\mu))] \right\} = \inf_q \{ \text{co}(\phi_B^*(q/\mu)) \} \geq \inf_q \{ (\phi_B^*)'(q/\mu) \} \geq \Lambda_B J_B(\ell_B),$$

using the fact that the derivative of the concave closure of a function is lower bounded by the infimum of the derivatives of that function.

Similarly, on the seller side we have

$$(\phi_S^*)'(q) = \Lambda_S \left[ \Delta_S(G_S^{-1}(q) - G_S^{-1}(q)) - (\Lambda_B - \Lambda_S) J_S(G_S^{-1}(q)) \right] \geq \Lambda_S \left[ \frac{G_S(G_S^{-1}(q))}{g_S(G_S^{-1}(q))} - G_S^{-1}(q) \right] - (\Lambda_B - \Lambda_S) J_S(G_S^{-1}(q)) = -\Lambda_B J_S(G_S^{-1}(q)) \geq -\Lambda_B J_S(\bar{r}_S),$$

using the assumption that virtual cost is monotone. Therefore,

$$\inf_q \{ \text{co}(\phi_S^*)'(q) \} \geq \inf_q \{ (\phi_S^*)'(q) \} \geq -\Lambda_B J_S(\bar{r}_S).$$

The obtained inequalities imply that the derivative of $\mu \text{co}(\phi_B^*)(Q/\mu) + \text{co}(\phi_S^*)(Q)$ is lower bounded by

$$\Lambda_B [J_B(\ell_B) - J_S(\bar{r}_S)] = \Lambda_B \left[ \ell_B - \bar{r}_S - \left( \frac{1}{g_B(\ell_B)} + \frac{1}{g_S(\bar{r}_S)} \right) \right],$$

which is non-negative by assumption of the theorem. Because the Lagrangian $\mu \text{co}(\phi_B^*)(Q/\mu) + \text{co}(\phi_S^*)(Q)$ is non-decreasing, the optimal volume of trade is equal to the maximal feasible quantity: $Q = \min\{\mu, 1\}$. Assume that $\mu > 1$ so that $Q = 1$.

To finish the proof, recall from the proof of Theorem 3 that, when $\alpha^* = \Lambda_B$ and buyer-side inequality is high, the function $\phi_B^*$ lies strictly below its concave closure when the fraction of buyers trading is sufficiently close to 1.\(^{37}\) Because the optimal volume of trade is 1 and the mass of buyers is $\mu$, when $\mu \in (1, 1 + \epsilon)$, the fraction of buyers trading in the optimal mechanism is arbitrarily close to 1 for small $\epsilon$. Thus, there exists $\epsilon > 0$ such that the optimal mechanism rations the buyers whenever $\mu \in (1, 1 + \epsilon)$ (rationing is equivalent to $\phi_B^*$ lying below its concave closure at the optimal volume of trade).

\(^{37}\)In the proof of Theorem 3, we normalized $\mu = 1$; thus, $q$ close to 1 in the proof of Theorem 3 should be interpreted as $q$ close enough to $\mu$ when $\mu$ is arbitrary.
B.9 Proofs of results in Section 3

Finally, we explain how the results stated in Section 3 follow from the general results stated in Sections 4 and 5.

First, note that while the one-sided problems considered in Section 3 are formally different from the two-sided problem studied in Sections 4 and 5, the techniques extend immediately to this case because most of our analysis looked at the two sides of the market separately. In particular, optimality of rationing on side \( j \) depends solely on the properties of the function \( \phi_{j}^{\alpha} \). This is still the case in the one-sided problem. The only differences are that (i) the budget constraint has an exogenous revenue level \( R \), and (ii) \( Q \) is fixed rather than determined endogenously. Thus, optimality of rationing depends on whether or not the function \( \phi_{j}^{\alpha} \) lies below its concave closure at the fixed quantity \( Q \).

Next, we note that under the assumption of uniform distribution, all of the functions \( G^{-1}(q) - \Delta_B(G^{-1}(q)) \), \( J_B(G^{-1}(q), G^{-1}(q) - \Delta_S(G^{-1}(q))) \), and \( J_S(G^{-1}(q)) \) are convex. By inspection of the proof of Theorem 3, this implies that regardless of the Lagrange multiplier \( \alpha \), the function \( \phi_{B}^{\alpha}(q) \) is first concave and then convex, and the function \( \phi_{S}^{\alpha}(q) \) is first convex and then concave. Consequently, we observe that there exists \( q_{\alpha}^{B} \) such that rationing on the buyer side is optimal if and only if \( Q \in (q_{\alpha}^{B}, 1) \) (with \( \mu \) normalized to 1). Similarly, there exists \( q_{\alpha}^{S} \) such that rationing on the seller side is optimal if and only if \( Q \in (0, q_{\alpha}^{S}) \).

B.9.1 Proof of Proposition 1

When seller-side inequality is low, the function \( \phi_{S}^{\alpha}(q) \) is strictly concave and thus a competitive mechanism is optimal.

When seller-side inequality is high, rationing is optimal if and only if \( Q \in (0, q_{\alpha}^{S}) \) for \( q_{\alpha}^{S} \) defined above. Moreover, by Theorem 4, whenever it is optimal to ration, it is optimal to ration at a single price. We can define \( Q(R) \) as \( q_{\alpha}^{S} \) with \( \alpha = \alpha_{R}^{*} \), being the optimal Lagrange multiplier on the budget constraint with revenue target \( R \). Then, to establish Proposition 1, it only remains to show the three properties of the function \( Q(R) \):

1. \( Q(R) \) is strictly positive for high enough \( R \); indeed, when \( R \) is high enough, sellers must receive a strictly positive lump-sum transfer in the optimal mechanism. But then, we must have \( \alpha_{R}^{*} = \Lambda_{S} \), and thus \( q_{\alpha}^{S} > 0 \), by the proof of Theorem 3.

2. \( Q(R) < 1 \) for all \( R \); this follows directly from Theorem 3.

3. \( Q(R) \) is increasing; this follows from two claims: First, the optimal Lagrange multiplier \( \alpha_{R}^{*} \) is decreasing in the revenue level \( R \) (a higher \( R \) corresponds to an easier-to-satisfy
constraint, so the corresponding Lagrange multiplier must be lower);\(^\text{38}\) Second, \(\phi_S^{\alpha_1} - \phi_S^{\alpha_2}\) is a concave function when \(\alpha_1 \geq \alpha_2\); thus, the set of points at which \(\phi_S^{\alpha_1}\) lies below its concave closure is contained in the set of points at which \(\phi_S^{\alpha_2}\) lies below its concave closure. It follows that \(q_S^x\) is decreasing in \(\alpha\). Putting the two preceding observations together, we conclude that \(Q(R)\) is increasing.

**B.9.2 Proof of Proposition 2**

Differentiating the designer’s objective function over \(p_B\) yields

\[
Q \left[ \frac{g_B(p_B)}{(1 - G_B(p_B))^2} \int_{p_B}^{\bar{r}_B} \lambda_B(r)(r - p_B) dG_B(r) + \Lambda_B - \frac{1}{1 - G_B(p_B)} \int_{p_B}^{\bar{r}_B} \lambda_B(r) dG_B(r) \right] \geq 0
\]

\[
\geq Q \left[ \Lambda_B - \int_{p_B}^{\bar{r}_B} \frac{\lambda_B(r) dG_B(r)}{1 - G_B(p_B)} \right] = Q \left[ \mathbb{E}^B[\lambda_B(r)] - \mathbb{E}^B[\lambda_B(r)| r \geq p_B] \right] \geq 0, \quad (B.22)
\]

where the last inequality follows from the fact that \(\lambda_B(r)\) is non-increasing (note that this inequality corresponds to the comparison of forces \((ii)\) and \((iii)\) described in the discussion of Proposition 2). This shows that the objective function of the designer is non-decreasing in the choice variable; thus, it is optimal to set \(p_B\) to be equal to its upper bound \(G_B^{-1}(1 - Q) = p_B^C\).

**B.9.3 Proof of Proposition 3**

The argument is fully analogous to the proof of Proposition 1.

**B.9.4 Proof of Proposition 4**

Proposition 4 is a special case of Theorem 2.

**B.9.5 Proof of Proposition 5**

Proposition 5 is a special case of Theorem 5; the conclusion that rationing happens at a single price follows from Theorem 4.

**B.9.6 Proof of Proposition 6**

Proposition 6 is a special case of Theorem 6.

\(^{38}\)Formally, this claim follows from analyzing the dual problem: The Lagrange multiplier is equal to the optimal dual variable in the dual problem; a lower constant \(R\) implies that the dual variable in the dual objective function is multiplied by a smaller positive scalar; thus the optimal \(\alpha^*_R\) cannot increase.
B.9.7 Proof of Proposition 7

Proposition 7 is a special case of Theorem 7. Note that here the constant $M$ in the proof of Theorem 7 is given by $M = 1/g_B(\bar{r}_B) + 1/g_S(\bar{r}_S)$. Specializing to the case of uniform distribution, we obtain that the condition $\bar{r}_B - \bar{r}_S \geq M$ is equivalent to

$$\bar{r}_B - \bar{r}_S \geq (\bar{r}_B - \bar{r}_B) + (\bar{r}_S - \bar{r}_S) \iff \bar{r}_B - \bar{r}_S \geq \frac{1}{2}(\bar{r}_B - \bar{r}_S),$$

which is the condition assumed in Proposition 7.