# Screening Adaptive Cartels

Juan Ortner
Boston University

Sylvain Chassang
Princeton University

Kei Kawai
U.C. Berkeley

Jun Nakabayashi*†
Kindai University

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**Abstract**

We propose an equilibrium theory of data-driven antitrust oversight in which regulators launch investigations on the basis of suspicious bidding patterns and cartels can adapt to the statistical screens used by regulators. We emphasize the use of asymptotically safe tests, i.e. tests that are passed with probability approaching one by competitive firms, regardless of the underlying economic environment. Our main result establishes that screening for collusion with safe tests is a robust improvement over laissez-faire. Safe tests do not create new collusive equilibria, and do not hurt competitive industries. In addition, safe tests can have strict bite, including unraveling all collusive equilibria in some settings. We provide evidence that cartel adaptation to regulatory oversight is a real concern.

KEYWORDS: collusion, auctions, bidding rings, cartels, procurement, antitrust.

*Contact information: Ortner: jortner@bu.edu, Chassang: chassang@princeton.edu, Kawai: kei@berkeley.edu, Nakabayashi: nakabayashi.1@eco.kindai.ac.jp.

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# 1 Introduction

Competition authorities commonly rely on statistical screens to detect and investigate collusion between firms.[1] Even if formal prosecution cannot rely on statistical evidence alone, detection tools can greatly facilitate the work of competition authorities. Such evidence can be used in court to obtain warrants or authorization for a more intrusive investigation, ultimately leading to actionable evidence and convictions (see Imhof et al., 2018, for a concrete example).[2] Furthermore, statistical evidence may be helpful in convincing cartel members to apply to leniency programs.[3] However, this growth in the use of statistical screens raises numerous questions: Do firms adapt to the statistical tests implemented by competition authorities? If they do, what is the impact of such screens in equilibrium? Can the tests backfire and either strengthen cartels, or hurt competitive firms? Can we find tests that do not cause harm, and yet reduce the incentives to form cartels? We provide theory and evidence addressing these questions.

We propose a model collusion in the shadow of antitrust authorities. A group of firms repeatedly participates in a first-price procurement auction. We allow firms to observe arbitrary signals about one another, and allow bidders' costs to be correlated within periods.[4] At a finite date $T$, the antitrust authority observes the history of bids placed by firms, and performs a screening test to determine whether or not they acted competitively. Firms

[1]Competition authorities that use statistical analysis or algorithms to screen for collusion include those in Brazil, South Korea, Switzerland and United Kingdom. A report by the OECD (2018) gives a brief description of the screening programs used in Brazil, Switzerland and the U.K. A document titled "Cartel Enforcement Regime of Korea and Its Recent Development" maintained by the Fair Trade Commission of Korea describes South Korea's bid screening program.

[2]Baker and Rubinfeld (1999) give an overview of the use of statistical evidence in court for antitrust litigation. In some jurisdictions, statistical evidence from screens have been used successfully to build a collusion case in court. See Mena-Labarthe (2015) for a case-study from Mexico.

[3]Screening for cartel behavior can also be useful to stakeholders other than antitrust authorities. For example, screening can help procurement offices counter suspected bidding rings by more aggressively soliciting new bidders or adopting auction mechanisms that are less susceptible to collusion. Screening may also be helpful for internal auditors and compliance officers of complicit firms to identify collusion and help contain potential legal risks arising from compliance failures.

[4]Our results extend as is if costs are correlated across periods via a publicly observed exogenous time-varying state.

that don't pass the test are further investigated, and may incur penalties if found guilty of bid-rigging. Investigation may also be costly to non-cartel members. We say that the tests used by the antitrust authority are asymptotically safe test if and only if competitive bidders pass with probability approaching one as data becomes large under all environments in the support of beliefs. Our companion papers Chassang et al. (2022) and Kawai et al. (2021) develop asymptotically safe tests and illustrate their relevance by applying them to procurement data from Japan and the US.

Our main set of results shows that antitrust oversight based on asymptotically safe tests is a robust improvement over laissez-faire. First, we show that regulation based on firm-level asymptotically safe tests does not significantly expand the set of enforceable collusive schemes available to cartels. Hence, asymptotically safe tests don't increase firms incentives to form a cartel. This addresses a concern raised by Cyrenne (1999) and Harrington (2004) that some natural screens against collusion may backfire and enhance the ability of cartels to collude. Second, we establish by example that asymptotically safe tests can have strict bite. In complete information settings, optimally colluding bidders submit nearly tied bids. In addition, testing for an excessive mass of close bids yields an asymptotically safe test whenever firms face strictly positive bid preparation costs. This safe test causes collusive equilibria to unravel for an open range of discount factors.

Our second contribution is to provide suggestive evidence that cartels do in fact adapt to regulatory screens. Tests of frequent close bids turn out to be similar to variance screens frequently used by regulators (Abrantes-Metz et al., 2005, Imhof et al., 2016): such screens flag auctions with low bid dispersion, i.e. auctions whose bids are unusually close together. If regulators scrutinize auctions with close bids, a cartel seeking to avoid scrutiny may coordinate its members to avoid placing close bids. This adaptive response may then lead to "missing bids" in large datasets: instead of being excessively frequent, adaptation would lead close bids to become excessively rare. Remarkably, we show that this pattern is present in procurement auctions from Japan. In addition, as we argue in Chassang et al. (2022), testing for missing bids also constitutes a safe test.[5]

Our paper relates to the academic literature on statistical screens to detect non-competitive behavior.[6] Collusive bidding patterns can be detected by measuring the level of correlation among bids (Bajari and Ye, 2003), by looking for price patterns predicted by the theory of repeated games (Porter, 1983, Ellison, 1994), or by exploiting changes in the auction format (Chassang and Ortner, 2019). Statistical tests of collusion have also been developed for average-price auctions (Conley and Decarolis, 2016) and multi-stage auctions with rebidding (Kawai and Nakabayashi, 2018). Our paper complements this literature by presenting evidence suggesting that cartels do adapt to regulatory screens, and by showing that statistical screens based on safe tests do not create new collusive equilibria.

A smaller literature studies the equilibrium impact of antitrust oversight. Besanko and Spulber (1989) and LaCasse (1995) study static models of equilibrium regulation. Closer to our work, Cyrenne (1999) and Harrington (2004) study repeated oligopoly models in which colluding firms might get investigated and fined whenever prices exhibit large and rapid fluctuations. Both papers highlight that antitrust oversight may backfire, allowing cartels to sustain higher equilibrium profits. Intuitively, cartels may use the threat of a regulatory crackdown to discipline their members.[7] We provide evidence that concerns about adaptive cartels are valid, but that they can be addressed using safe tests.

Our work complements the literature on auction design in the presence of collusion. Abdulkadiroglu and Chung (2003), Che and Kim (2006, 2009) and Pavlov (2008) show that appropriate auction design can limit the cost of collusion when cartel members have deep pockets and can make payments upfront. Che et al. (2018) studies optimal auction design when collusive bidders are cash-constrained. Our paper complements this literature

[5]In Chassang et al. (2022) we also show that missing bids are correlated with plausible markers of collusion: missing bids are more prevalent in auctions with high winning bids, and, for industries that are investigated for bid-rigging, they are more prevalent before the investigation than after.

[6]See Porter (2005) and Harrington (2008) for recent surveys of this literature.

[7]See also McCutcheon (1997), who shows that anti-trust oversight may help sustain collusion by reducing firms' incentives to renegotiate equilibrium play after a deviation.

by showing how an antitrust agency can limit the impact of collusion by screening firms using safe tests.

The paper is structured as follows. Section 2 sets up our model of collusion in the shadow of investigation. Section 3 presents an example in which a naïve test for collusion ends up strengthening cartels. Section 4 introduces safe tests. Section 5 establishes that safe tests do not create new collusive equilibria and can strictly reduce the payoffs of cartels. Section 6 provides evidence that cartels do in fact adapt to statistical screens of collusion. Section 7 concludes. Proofs are collected in Appendix A.

# 2 Colluding in the Shadow of Antitrust Authorities

We model the interaction between cartel members and antitrust authorities as follows. At each period $t \in \mathbb{N}$, firms participate in a procurement auction. At some fix time $T \in \mathbb{N}$, the regulator applies tests to the data generated by the players in periods $t \leq T$. If a test comes out against the null hypothesis of competition, one or more firms are investigated. We begin by describing the stage game of the repeated game.

## 2.1 Repeated Procurement

In each period $t \in \mathbb{N}$, a buyer needs to procure a single project from a finite set $N = \{1, ..., n\}$ of potential suppliers. The auction format is a sealed-bid first-price auction with reserve price $r$, which we normalize to $r = 1$. Let $B \subset [0, 1]$ be the set of feasible bids. We assume throughout that $B = [0, 1]$, except in Proposition 2, where we assume $B = [0, \nu, 2\nu, ..., 1]$, with $\nu > 0$ small.

**Costs.** Firms' period $t$ procurement costs are denoted by $\mathbf{c}_t = (c_{i,t})_{i \in N}$. For simplicity, we assume private values, so that each bidder observes her own cost. Costs are allowed to be correlated across firms within each period, but for simplicity are assumed to be i.i.d. over time: at each time $t$, $\mathbf{c}_t$ is drawn from distribution $F_C(\cdot)$ supported on set $C \subset [0,1]^n$.[8]

**Information.** In each period $t$, each bidder $i \in N$ privately observes a signal $z_{i,t}$ prior to bidding. Signals $z_{i,t}$ can take arbitrary values, including vectors in $\mathbb{R}^k$. The distribution of signals $\mathbf{z}_t = (z_{i,t})_{i \in N}$ depends only on realized costs $\mathbf{c}_t$: $\mathbf{z}_t$ is drawn from distribution $F_Z(\cdot|\mathbf{c}_t)$, with support contained in some set $Z$. Signals $(z_{i,t})_{i \in N}$ are allowed to be arbitrary, and may include information about the costs of other bidders. This allows our model to nest many informational environments, including correlated private values, asymmetric bidders, as well as complete information. Since bidders observe their own costs prior to bidding, we assume that firm $i$'s signal $z_{i,t}$ includes firm $i$'s cost $c_{i,t}$.

**Bids.** After privately observing signal $z_{i,t}$, each firm $i \in N$ submits a bid $b_{i,t} \in B \cup \emptyset$, where $\emptyset$ denotes not participating. The procurement contract is allocated to the bidder submitting the lowest bid in $B$, at a price equal to her bid. Ties are broken randomly. We assume that each bidder $i \in N$ incurs a bid preparation cost $\kappa_i \geq 0$ from submitting a bid in $B$. Profiles of bids are denoted by $\mathbf{b}_t = (b_{i,t})_{i \in N}$, with $\wedge \mathbf{b}_t$ denoting the lowest bid. We let $\mathbf{b}_{-i,t} \equiv (b_{j,t})_{j \neq i}$ denote bids from firms other than firm $i$, and define $\wedge \mathbf{b}_{-i,t} \equiv \min_{j \neq i} b_{j,t}$ to be the lowest bid among $i$'s competitors.

We assume that bids are publicly revealed at the end of each period. This matches standard practices in public procurement, where legislation typically requires governments to make bids public. Our main results can be adapted if only the winning bid is made public, or if bidders only observe the identity of the winner.

Overall, firm $i$'s profits in period $t$ are

$$\pi_{i,t} = x_{i,t} \times (b_{i,t} - c_{i,t}) - \kappa_i \mathbf{1}_{b_{i,t} \neq \emptyset},$$

where $x_{i,t} \in \{0,1\}$ denotes whether or not firm $i$ wins the auction at time $t$. Firms discount

[8] Our analysis remains unchanged if we allow costs to depend on a public time-varying exogenous state.

future payoffs using common discount factor $\delta < 1$.

## 2.2 Antitrust Oversight

We assume that the antitrust authority runs screening tests at period $T$, based on data from the $M$ periods leading up to $T$.

Fix $T \in \mathbb{N}$ and $M \in \mathbb{N}$, $M \leq T+1$: $T$ is the testing date, and $M$ is the length of the monitoring phase. Let $h_{M,T} = (\mathbf{b}_s)_{s=T+1-M}^{T}$ denote the bids placed during the monitoring phase $t = T+1-M, ..., T$. At the end of period $T$, after players placed their bids and the auction's outcome is realized, the antitrust authority runs a vector of tests $(\tau_i)_{i \in N}$, with $\tau_i : h_{M,T} \mapsto \tau_i(h_{M,T}) \in \{0,1\}$.[9] If test $\tau_i$ takes value 1, firm $i$ is investigated. This yields an expected penalty $K_T \geq 0$, paid at period $T$. We allow penalty $K_T$ to grow as the testing date $T$ grows large: for instance, we allow for $K_T = \delta^{-T} K$ for some $K > 0$. Hence, the impact of antitrust oversight on firms payoffs at the start of the game may remain bounded away from zero even as $T$ grows large.

For simplicity, we consider fixed penalties. However, we note that all of our results would continue to hold if penalty $K_T$ was allowed to depend on bidding history $h_{M,T}$. This would allow for penalties that depend on the extent, or impact, of collusion.

Aggregate payoffs to firm $i$ from the perspective of period 0 take the form

$$(1-\delta)\left[\sum_{t=0}^{\infty} \delta^t \pi_{i,t}\right] - \tau_i \delta^T K_T.$$

**Solution concept.** The period-$t$ public history $h_t$ takes the form $h_t = (\mathbf{b}_s)_{s<t}$. Because costs are drawn i.i.d. across periods, past play conveys no information about the private types of other players. As a result we do not need to specify out-of-equilibrium beliefs. A

[9]More generally, $h_{M,T}$ may include any data observable to the antitrust authority at the time of running the test.

public strategy $\sigma_i$ is a mapping

$$\sigma_i : h_t, z_{i,t} \mapsto b_{i,t}.$$

Under a public strategy, firm $i$'s bid at each time $t$ depends on public history $h_t$ and current signal realization $z_{i,t}$. We focus on perfect public Bayesian equilibria (Athey and Bagwell, 2008); i.e., perfect Bayesian equilibria in public strategies.

# 3 A Motivating Example

In prior work, Cyrenne (1999) and Harrington (2004) showed that regulatory oversight may backfire, allowing cartels to sustain larger profits. We now illustrate this possibility in the context of our model.

We assume that procurement costs are publicly observed and equal to zero: $\forall i, t$, $c_{i,t} = 0$. We also assume that there are no participation costs: $\forall i$, $\kappa_i = 0$.

Fix $\eta \in (0, 1)$. Suppose the regulator runs the tests at period $T$, with monitoring length $M = T+1$. Hence, the outcome of the test depends on bidding behavior at periods $t = 0, .., T$. Suppose further that the regulator runs the same test for all firms in the industry: for all $i \in N$, $\tau_i = \tau^{\mathsf{break}}$ with $\tau^{\mathsf{break}}$ defined by

$$\tau^{\mathsf{break}} \equiv \mathbf{1} \left\{ \exists S \le T \text{ s.t. } |\wedge \mathbf{b}_t - \wedge \mathbf{b}_s| > \eta \text{ for all } t < S, s \in [S, T] \right\}.$$

Test $\tau^{\mathsf{break}}$ looks for structural breaks in bidding behavior: firms fail the test if there is a discrete jump in the winning bid. Testing for structural breaks is a common way of screening for cartels (e.g., Harrington, 2008): failure to pass test $\tau^{\mathsf{break}}$ might indicate either that a cartel was formed or that a cartel collapsed.

Assume $n \ge 2$, $\eta < \frac{1}{n}$ and $\delta < \frac{n-1}{n}$. Consider first the case with no regulator; i.e. $K_T = 0$. Since $\delta < \frac{n-1}{n}$, firms are unable to sustain supra-competitive prices. In any equilibrium the winning bid must be equal to zero at all periods: $b_{(1),t} = 0$ for all $t$.

Consider next the case with a regulator. When $K_T$ is sufficiently large, each player $i \in N$ playing according to the following strategy constitutes an equilibrium of the regulatory game:

- at the initial history $h_0$, or at any history $h_t$ with $t < T$ and with $b_{j,s} = 1$ for all $j \in N, s < t$, bid $b_{i,t} = 1$;
- at history $h_T$ with $b_{j,s} = 1$ for all $j \in N, s < T$, bid $b_{i,t} = 1 - \eta$;
- at any other history $h_t$, bid $b_{i,t} = 0$.

Intuitively, play reverts to static Nash following any deviation prior to period $T$, leading firms to fail the test. When penalty $K_T$ is sufficiently large, the loss from failing the test outweighs any deviation gain.[10]

# 4 Safe Tests

The previous example illustrates that screening for collusion may inadvertently strengthen cartels. This section introduces a class of tests, which we call *asymptotically safe tests*. In words, asymptotically safe tests are tests that are passed with probability close to one by unilaterally competitive firms. Our main results, presented in Section 5, show that preventing harm against competitive firms serves both a direct purpose – it has a vanishingly small impact on firms operating in a competitive market – and an indirect one – it ensures that regulatory screens do not increase a cartel's enforcement capability. Altogether this implies that safe tests do not increase firms' incentives to cartelize.

Before introducing safe tests, we need to define what we mean by competitive firms. Following Chassang et al. (2022), we say that a firm is competitive if and only if it plays a stage-game best response at every history on the equilibrium path. Recall that $T$ is the

[10]Note that, under this strategy profile, if there are no deviations prior to $T$, firms only have an incentive to undercut the winning bid $b_{(1),T}$ at $T$ when $b_{(1),T} > 1 - \eta$. Indeed, firms fail the test if the winning bid at $T$ falls below $1 - \eta$.

period at which the tests are run. Periods $t = T+1-M, ..., T$ correspond to the monitoring phase, and $h_{M,T} = (\mathbf{b}_s)_{s=T+1-M}^{T}$ are the bids placed during the monitoring phase.

**Definition 1** (competitive histories)**.** *Fix a common knowledge profile of play $\sigma$ and a history $h_{i,t} = (h_t, z_{i,t})$ of player $i$. Firm $i$ is competitive at history $h_{i,t}$ if play at $h_{i,t}$ is stage-game optimal for firm $i$ given the behavior of other firms $\sigma_{-i}$.*

*Firm $i$ is competitive during the monitoring phase if it plays competitively at all on-path histories $h_{i,t}$ with $t = T+1-M, ...., T$.*

*Firm $i$ is competitive if it plays competitively at all histories on the equilibrium path.*

*The industry is competitive if all firms $i \in N$ play competitively at all histories on the equilibrium path.*

We note that, if the industry is competitive under $\sigma$, firms must be playing a stage-game equilibrium in every period along the path of play.

**Asymptotically safe tests.** The economic environment corresponds to the tuple $E = (F_C, F_Z, (\kappa_i)_{i\in N})$. We denote by $\mathcal{E}$ the set environments $E$ that the antitrust authority deems feasible. Set $\mathcal{E}$ captures the subjective restrictions that the antitrust authority is willing to place.

**Definition 2** (asymptotically safe tests)**.** *Tests $(\tau_i)_{i\in N}$ are asymptotically safe if and only if for all $i \in N$, all $E \in \mathcal{E}$, and all profiles $\sigma$ such that firm $i$ is competitive during the monitoring phase, $\mathsf{prob}_{E,\sigma}(\tau_i = 0) \geq 1 - G(M)$ for some $G(\cdot) \geq 0$ with $\lim_{S\to\infty} G(S) = 0$.*

In words, tests $(\tau_i)_{i\in N}$ are asymptotically safe if they admit a vanishingly small rate of false positives. This concern over false positives coincides with concerns expressed by regulators (Imhof et al., 2016). In practice, investigation is a highly disruptive process that is only triggered if sufficiently many pieces of evidence are collected.[11] Our companion

[11] This is not to say that a regulator would not launch an investigation on the basis of somewhat imperfect evidence. Rather, that there is little cost in using safe tests.

papers Chassang et al. (2022) and Kawai et al. (2021) propose various asymptotically safe tests and illustrate their relevance by applying them to Japanese procurement data, as well as data from the Ohio school milk cartel (Porter and Zona, 1999). We emphasize that asymptotically safe tests pass competitive firms with high probability even if other firms do not behave competitively.

The function $G(\cdot)$ in Definition 2 bounds the rate of false positives of tests $(\tau_i)_{i \in N}$. In Chassang et al. (2022) we propose a family of asymptotically safe tests whose outcomes are based on estimated demand, and show that tests in this family have a rate of false positive that is exponentially decreasing in the number of observations: i.e., those tests are asymptotically safe with $G(M) = \exp(-\alpha M)$ for some constant $\alpha > 0$. In Section 5.3 we present an asymptotically safe test that also has an exponentially decreasing rate of false positives.

# 5 Safe Tests do not Increase Incentives to Cartelize

This section provides normative foundations for safe tests. We proceed in three steps: first, we show that safe tests do not significantly expand the set of (potentially collusive) equilibria; second, we show that safe tests do not significantly affect competitive industries; third, we show by means of an example that safe tests can have strict bite, including the unraveling of all collusive equilibria.

## 5.1 Safe Tests do not Create New Collusive Equilibria

We start by showing that asymptotically safe tests do not significantly enlarge the set of equilibria. We formalize this with two results. Our first result establishes a bound on the equilibrium payoff set of the game with a regulator that holds for any testing date $T$ and monitoring length $M$. Our second result shows that any equilibrium of the game with the regulator must be "close" to an equilibrium of the game without the regulator when testing date $T$ and monitoring length both $M$ grow large.

**Bound on the equilibrium payoff set.** For each environment $E \in \mathcal{E}$, and integers $T$ and $M \leq T+1$, let $\Sigma_{T,M}(E)$ denote the set of equilibria of the game with testing date $T$ and monitoring length $M$. Let $\Sigma(E)$ denote the set of equilibria of the game without the regulator.

Fix an environment $E \in \mathcal{E}$, a testing date $T$ and a monitoring length $M \leq T+1$. For each strategy profile $\sigma$ and each history $h_{i,t}$, let $V_i^E(\sigma, h_{i,t})$ denote player $i$'s expected discounted payoff (excluding possible penalties) at history $h_{i,t}$ under strategy profile $\sigma$ and environment $E$. Similarly, let $P_i^E(\sigma, h_{i,t}) = \mathbf{1}_{t \leq T} \mathbb{E}_{E,\sigma}\left[\tau_i \delta^{T-t} K_T | h_{i,t}\right]$ denote player $i$'s expected penalty at history $h_{i,t}$ under strategy profile $\sigma$ and environment $E$. Player $i$'s total payoff in the regulatory game is $W_i^E(\sigma, h_{i,t}) = V_i^E(\sigma, h_{i,t}) - P_i^E(\sigma, h_{i,t})$.

For $T$ and $M$, define

$$\mathcal{V}_{T,M}(E) \equiv \left\{ \mathbf{V} \in \mathbb{R}^n : \mathbf{V} = \left( \mathbb{E}_E[W_i^E(\sigma, (h_0, z_i))] \right)_{i \in N} \text{ for some } \sigma \in \Sigma_{T,M}(E) \right\},$$

to be the set of payoffs that can supported in an equilibrium with testing date $T$ and monitoring length $M$ under environment $E$. Similarly, let

$$\mathcal{V}(E) \equiv \left\{ \mathbf{V} \in \mathbb{R}^n : \mathbf{V} = \left( \mathbb{E}_E[V_i^E(\sigma, (h_0, z_i))] \right)_{i \in N} \text{ for some } \sigma \in \Sigma(E) \right\}$$

denote the equilibrium payoff set in of the game without a regulator.

Let $X = B \cup \emptyset$ denote the set of bids and fix $\mathcal{W} \subset \mathbb{R}^n$. A profile of bidding functions $\beta = (\beta_i)_{i \in N}$, with $\beta_i : z_i \mapsto \beta_i(z_i) \in \Delta(X)$ for each $i \in N$, is enforceable on $\mathcal{W}$ if there exists $W : X^n \to \mathcal{W}$ such that, for all $i \in N$, for all $z_i$, and all $b_i \in \mathsf{supp}\, \beta_i(z_i)$,

$$b_i \in \arg\max_b \mathbb{E}_{E,\beta}[(1-\delta)(b - c_i)x_i + \delta W_i(b, \mathbf{b}_{-i}) | z_i],$$

where $x_i \in \{0, 1\}$ denotes whether or not $i$ wins the auction.

Fix a profile of bidding functions $\beta$ enforced by $W$ on $\mathcal{W}$. For each $i \in N$, define

$$v_i^E(\beta, W) \equiv \mathbb{E}_{E,\beta}[(1-\delta)(b_i - c_i)x_i + \delta W_i(b_i, b_{-i})].$$

Hence, payoffs $(v_i^E(\beta, W))_{i \in N}$ are decomposable on $\mathcal{W}$ under environment $E$. For any $\mathcal{W} \subset \mathbb{R}^n$, the APS operator is given by

$$\mathcal{B}_E(\mathcal{W}) \equiv \{\mathbf{V} \in \mathbb{R}^n : \mathbf{V} = (v_i^E(\beta, W))_{i \in N} \text{ for some } \beta \text{ enforced by } W \text{ on } \mathcal{W}\}.$$

For each monitoring date $T$ and length of monitoring phase $M$, define $\widehat{\mathcal{V}}_{T,M} \equiv [-G(M)\delta^M K_T, 1]^n$.

**Proposition 1.** *Suppose the regulator runs asymptotically safe tests. Then, for all environments $E \in \mathcal{E}$, $\mathcal{V}_{T,M}(E) \subset \mathcal{B}_E^{T-M}(\widehat{\mathcal{V}}_{T,M})$.*

In words, we can bound the set of equilibrium values of the game with the regulator by applying the APS operator $T - M$ times to the set $\widehat{\mathcal{V}}_{T,M}$. Recall that, in the game without a regulator, we have that $\mathcal{V}(E) = \lim_{t\to\infty} \mathcal{B}_E^t([0, 1]^n)$. Then, provided operator $\mathcal{B}_E$ is continuous, we have that $\mathcal{V}_{T,M}(E)$ will be "close" to $\mathcal{V}(E)$ as $T$ and $M$ grow large.

The key intuition behind Proposition 1 is that screening for collusion using tests that are asymptotically safe does not significantly lower firms' min-max values relative to a setting without a regulator. Indeed, following a history $h_{i,t}$ with $t \leq T - M$, a firm can guarantee to pass the test with probability $1 - G(M)$ by playing a static best response to the actions of her opponents during the monitoring phase. Hence, firm $i$'s continuation payoff at $t + 1$ can't be lower than $-G(M)\delta^{T-t}K_T$.

**Convergence of the equilibrium set.** Next, we show that, for all $E \in \mathcal{E}$, the equilibrium set $\Sigma_{T,M}(E)$ converges $\Sigma(E)$ as $T$ and $M$ grow large when the regulator runs asymptotically safe tests. To establish our result, we make the following assumptions. First, we assume that the set of feasible bids $B$, the set of possible cost profiles $C$, and the set of possible signal profiles $Z$ are all finite. This assumption guarantees that the set of mixed strategy profiles is compact under the product topology (by Tychonoff's Theorem).

Second, we assume that the length $M$ of the monitoring phase grows together with $T$, but at a slower rate: $M_T \sim O(T^x)$ for some constant $x \in (0,1)$. We let $(\tau_i^T)_{i\in N}$ denote the tests that the antitrust agency runs when the testing date is $T$ and the length of the monitoring period is $M = M_T$. Hence, when $(\tau_i^T)_{i\in N}$ are asymptotically safe, the probability of a false positive vanishes as $T$ grows large.

Lastly, for each testing date $T$, we assume that firms that fail the test incur a penalty $K_T = \delta^{-T}K$ for some $K > 0$. This implies that antitrust oversight continues to have an impact on firms' overall payoffs as $T$ grows.[12]

Endow the set of strategy profiles with the product topology, and for each $E \in \mathcal{E}$, define

$$\Sigma_\infty(E) \equiv \{\sigma : \exists (T^s) \to \infty, (\sigma^s) \to \sigma, \text{ with } \sigma^s \in \Sigma_{T^s,M^s}(E) \text{ for all } s\}$$

to be the set of limiting PPBE under environment $E$ as $T$ and $M$ grow to infinity.

**Proposition 2.** *Suppose the regulator runs asymptotically safe tests. Then, for all $E \in \mathcal{E}$, $\Sigma_\infty(E) \subset \Sigma(E)$.*

Fix $E \in \mathcal{E}$ and a sequence $(T^s) \to \infty$. Let $(\sigma^s)$ be a sequence satisfying $\sigma^s \in \Sigma_{T^s,M^s}(E)$ for all $s$. Then, by Proposition 2, $(\sigma^s)$ must approach the equilibrium set $\Sigma(E)$ as $s \to \infty$, even if the $(\sigma^s)$ does not converge.[13] Hence, Proposition 2 shows that antitrust oversight based on tests that are approximately safe (i.e., large $T$) does not significantly enlarge the set of enforceable collusive schemes available to cartels.

[12]Our results continue to hold as long as $K_T \leq \delta^{-T}K$.

[13]To see why, suppose the result is not true. Then, there exists a subsequence $(\sigma^{s_k})$ such that all elements in the subsequence are bounded away from $\Sigma(E)$. Since the set of strategy profiles is compact, $(\sigma^{s_k})$ has a convergent subsubsequence $(\sigma^{s_{k_m}})$. And by Proposition 2, $(\sigma^{s_{k_m}})$ converges to a point in $\Sigma(E)$. But this is a contradiction to $(\sigma^{s_k})$ being such that all its elements are bounded away from $\Sigma(E)$.

The following result shows that, in any equilibrium in $\Sigma_{T,M}(E)$, players expect to pass the test with high probability whenever $T$ is sufficiently large.

**Corollary 1.** *Suppose the regulator runs asymptotically safe tests, with testing date $T$ and a monitoring length $M$. Then, for all $E \in \mathcal{E}$, all $\sigma \in \Sigma_{T,M}(E)$, all $i \in N$ and all histories $h_{i,T-M}$,*

$$\mathsf{prob}_{E,\sigma}(\tau_i = 1 | h_{i,T-M}) \leq \frac{1}{\delta^{-(T-M)} K} + G(M).$$

*In particular, if $M \sim O(T^x)$ for some $x \in (0,1)$, $\mathsf{prob}_{E,\sigma}(\tau_i = 1 | h_{i,T-M}) \to 0$ as $T \to \infty$.*

## 5.2 Safe Tests do not Affect Competitive Equilibria

Let $\Sigma^{\mathsf{comp}}(E)$ denote the set of competitive equilibria under environment $E$: i.e., strategy profiles under which all bidders play according to the same Bayes' Nash equilibrium of the stage game at all histories. Assume that $\Sigma^{\mathsf{comp}}(E)$ is non-empty.[14] Then, the game with testing date $T$ and monitoring length $M$ has an equilibrium in which players play according to a Bayes' Nash equilibrium of the stage game at all periods $t \leq T - M$ (regardless of whether the tests that the regulator runs are safe or not). In addition, if the tests are asymptotically safe, we have that for all $E \in \mathcal{E}$, players' payoffs from using strategy profile $\sigma \in \Sigma^{\mathsf{comp}}(E)$ are bounded below by $\mathbb{E}_E[V_i^E(\sigma, h_0, z_i)] - G(M)\delta^T K_T$, where for any strategy profile $\hat{\sigma}$, $V_i^E(\hat{\sigma}, h_0)$ denotes player $i$'s payoffs at the start of the game (excluding penalties) when firms play according to $\hat{\sigma}$.

The following Proposition summarizes this discussion.

**Proposition 3.** *(i) For any tests $(\tau_i)$ and every $E \in \mathcal{E}$, the regulatory game has an equilibrium that coincides with an equilibrium in $\Sigma^{\mathsf{comp}}(E)$ for all periods $t \leq T - M$;*

*(ii) If tests $(\tau_i)$ are asymptotically safe, then for all $E \in \mathcal{E}$ and all $i \in N$, firm $i$'s payoffs from strategy profile $\sigma \in \Sigma^{\mathsf{comp}}(E)$ are bounded below by $V_i^E(\sigma, h_0) - G(M)\delta^T K_T$.*

[14] That is, assume that the stage game has a Bayes' Nash equilibrium.

## 5.3 Safe Tests Can Have Strict Bite

This subsection studies a benchmark setting in which procurement costs are publicly observable among firms. Our goal is to show that optimal collusive schemes in this setting fail natural safe tests that are frequently used by antitrust agencies. Moreover, as we show in Section 6, adaptive responses to these safe tests can help rationalize puzzling bidding patterns observed in Japanese procurement auctions.

For our results in this section, we make the following assumption:

**Assumption 1** (costly bid preparation)**.** *For all feasible environments $E = (F_C, F_Z, (\kappa_i)) \in \mathcal{E}$, $\kappa_i \geq \hat{\kappa} > 0$ for all $i \in N$.*

In words, under all plausible environments, firms face strictly positive bid-preparation costs. We note that bid preparation costs in public procurement can be substantial. For instance, Krasnokutskaya and Seim (2011) study highway procurement auctions in California, and estimate that bid preparation costs are between 2.2% and 3.9% of the engineer's cost estimate.

**A safe test.** Next, we introduce asymptotically safe tests ($\tau_i^{\mathsf{close}}$). Fix $\rho \in (0,1)$ and $\Delta \in (0,1)$. For each $i \in N$, define test $\tau_i^{\mathsf{close}}$ as:

$$\tau_i^{\mathsf{close}} \equiv \mathbf{1}\left(\frac{1}{M}\sum_{t=T+1-M}^{T} \mathbf{1}_{\wedge \mathbf{b}_{-i,t} \in (b_{i,t}-\Delta, b_{i,t}]} > \rho\right).$$

Test $\tau_i^{\mathsf{close}}$ looks for frequent close bids during the monitoring phase. Our next result shows that, under Assumption 1, test $\tau_i^{\mathsf{close}}$ is asymptotically safe.

**Proposition 4.** *Suppose Assumption 1 holds. Then, tests ($\tau_i^{close}$) with $\rho > \frac{\Delta}{\hat{\kappa}}$ are asymptotically safe: for all $E \in \mathcal{E}$, all $i \in N$, and all $\sigma$ such that $i$ is competitive during the monitoring phase, $\mathsf{prob}_{E,\sigma}(\tau_i^{close} = 0) \geq 1 - e^{-\frac{\alpha^2}{2}M}$ for some constant $\alpha > 0$.*

In words, when bid preparation costs are strictly positive, nearly-tied bids and winning bids near the reserve price should be infrequent under competitive behavior. We stress that regulators frequently use variance screens (Abrantes-Metz et al., 2005, Imhof et al., 2016) that flag auctions whose bids are unusually close together. This corresponds to the pattern of bids captured by test $\tau_i^{\mathsf{close}}$, with the adjustment that it focuses on the distance between firms' bids, rather than the variance of the bids. The match is exact when there are only two bidders.

**Optimal collusion in a benchmark setting.** We now show that optimal collusive schemes in benchmark settings fail tests $(\tau_i^{\mathsf{close}})$. Consider a special case of the repeated procurement model of Section 2 with two firms: i.e., $n = 2$. Consider a benchmark environment $E^{\mathsf{bmk}} \in \mathcal{E}$ such that, at each time $t$, firms share the same procurement cost $c$, which we normalize to $c = 0$. Suppose further that, under $E^{\mathsf{bmk}} \in \mathcal{E}$, each firm $i = 1, 2$ faces bid preparation cost $\kappa_i = \kappa > 0$.

We assume that the set of environments $\mathcal{E}$ that the regulator deems feasible satisfies Assumption 1, and that the antitrust authority uses tests $(\tau_i^{\mathsf{close}})$. Lastly, we assume that the auctioneer only runs the auction if both firms participate. This assumption makes bidding profiles in which firms alternate in participating suboptimal for the cartel.[15]

Let $\overline{V}^{\mathsf{bmk}}$ denote the largest cartel equilibrium payoff in the game without a regulator under environment $E^{\mathsf{bmk}}$. For each $T, M$, let $\overline{V}_{T,M}^{\mathsf{bmk}}$ denote the largest cartel equilibrium payoff (including potential penalties) in the game with testing date $T$ and monitoring phase $M$. Define $\underline{\delta} \equiv \frac{1}{1-\kappa}\frac{1}{2}$.

**Proposition 5.** *(i) Suppose there is no regulator. If $\delta \geq \underline{\delta}$, under an optimal equilibrium of the benchmark model both firms submit a bid equal to $r = 1$ at all on-path histories; i.e., $\overline{V}^{\mathsf{bmk}} = 1 - 2\kappa$.*

[15] Our results would continue to hold if the auctioneer runs the auction with probability $q < (1-2\kappa)/(1-\kappa)$ if only one firm participates.

*(ii) Suppose the regulator uses tests ($\tau_i^{\mathsf{close}}$). Then there exists $\mu > 0$ such that, for all $\varepsilon > 0$ and all $\delta \in [\underline{\delta}, \underline{\delta} + \mu]$, $\overline{V}_{T,M}^{\mathsf{bmk}} < \varepsilon$ whenever $M$, $T - M$ and $K$ are all large enough.*

Proposition 5(i) shows that, when there is no regulator and $\delta$ is higher than $\underline{\delta}$, optimal collusion involves both firms submitting a bid equal to the reserve price at all histories. Clearly, firms $i = 1, 2$ fail test $\tau_i^{\mathsf{close}}$ under this bidding profile. Proposition 5(ii) establishes that, if $\delta$ is not much larger than $\underline{\delta}$, firms are unable to sustain supra-competitive profits. Intuitively, the winning bid must be strictly below the reserve price a positive fraction of periods during the monitoring phase for firms to pass test $\tau_i^{\mathsf{close}}$, lowering equilibrium values. When $\delta$ is not too large, this lower equilibrium values preclude firms from sustaining supra-competitive profits.

# 6 Evidence of Cartel Adaptation

Safe tests ($\tau_i^{\mathsf{close}}$) rule out plausible collusive equilibria, and are in fact applied by regulatory agencies. If cartels adapt to regulatory screens, they may coordinate their bidding behavior to avoid triggering these particular tests. In fact, if a cartel is especially careful about not triggering these tests – more so than a competitive industry with nothing to fear would be – they might take a margin of safety generating noticeable patterns in large samples.

In particular, under tests ($\tau_i^{\mathsf{close}}$), cartel bidders may avoid bids such that $b^{(2)} - b^{(1)} < \epsilon$ and $r - b^{(1)} < \epsilon$ for $\epsilon > 0$ small. This means that in the sample of auction bids, the mass of close winning bids and the mass of winning bids near the reserve price will both be low.

Formally, for each auction $t$ and each bidder $i$ participating in this auction, let $\Delta_{i,t} \equiv (b_{i,t} - \wedge \mathbf{b}_{-i,t})/r_t$ denote the margin by which $i$ wins or losses the auction, and let $\wedge \mathbf{b}_t \equiv \min_i b_{i,t}$ denote the winning bid in the auction. A cartel that is trying to avoid triggering tests ($\tau_i^{\mathsf{close}}$) may generate a sample of bids such that the density of $\Delta_{i,t}$ is close to 0 around $\Delta_{i,t} = 0$, and such that the distribution of winning bids $\wedge \mathbf{b}_t$ has almost no mass close to the reserve price.

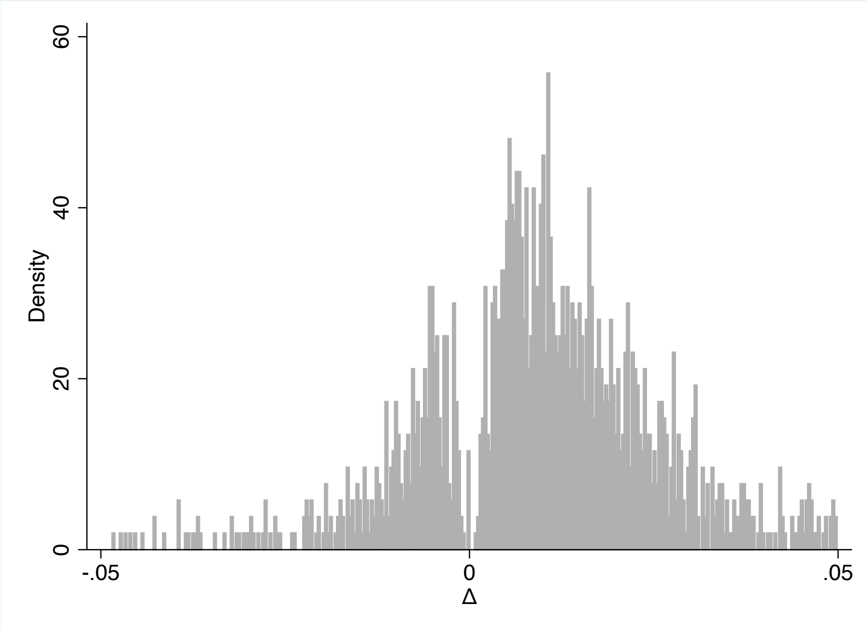


Figure 1: Distribution of bid differences $\Delta$

Both patterns are present in a sample of procurement auctions taking place in the city of Tsuchiura, in Ibaraki prefecture, Japan. The data contains approximately 400 auctions taking place between May 2007 and October 2009. The auction format is first-price sealed-bid, with a public reserve price. The median number of bidders is 5, and the median winning bid is approximately USD 98,000. In previous work, Chassang and Ortner (2019) provide evidence of bidder collusion in these auctions.

Figure 1 illustrates the distribution of normalized bid differences $\Delta_{i,t}$ for the sample of procurement auctions from Tsuchiura.[16] Figure 2 plots the empirical c.d.f. of normalized winning bids $\wedge \mathbf{b}_t / r_t$ (i.e., winning bid divided by reserve price) for this same sample of auctions. As anticipated, firms seek to avoid suspiciously close bids and winning bids close to the reserve price.[17]

[16]In Chassang et al. (2022), we show that similar bidding patterns also appear in procurement auctions run by the Ministry of Land, Infrastructure and Transportation in Japan, and in procurement auctions run by municipalities located in the Tohoku region in Japan.

[17]Kawai et al. (2022) proposes an alternative explanation for the bidding patterns in Figure 1. In particu-

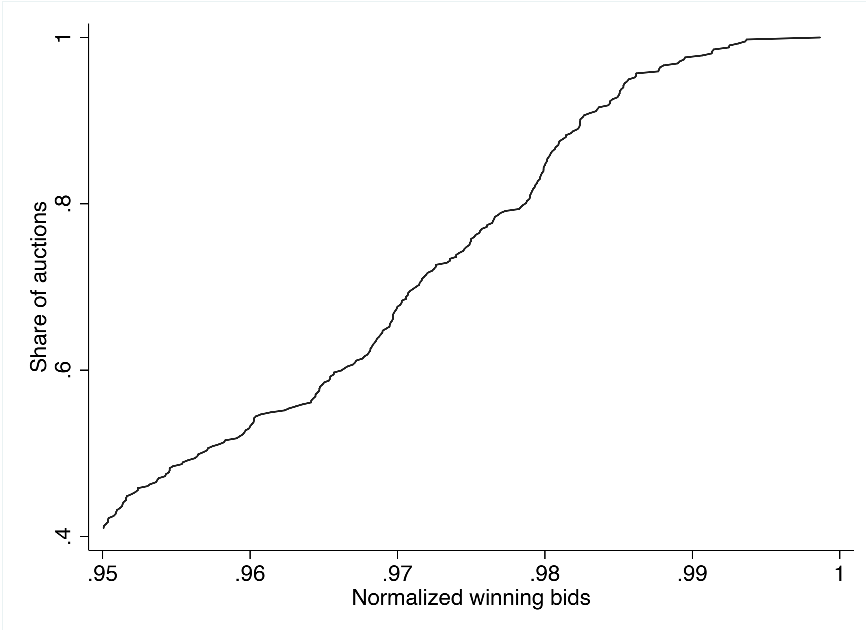


Figure 2: Cumulative distribution function of normalized winning bids

Interestingly, as we show in Chassang et al. (2022), this missing mass of bids around $\Delta = 0$ is itself a suspicious pattern that can be turned into a safe test. Recall that, for each bidder $i$ participating in auction $t$, $\wedge \mathbf{b}_{-i,t} = \min_{j \neq i} b_{j,t}$ is the lowest bid among $i$'s opponent in the auction. For each bidder $i \in N$, integers $T, M \leq T+1$ and constant $\rho \in (-1, \infty)$, let

$$\widehat{D}_i(\rho, T, M) \equiv \frac{1}{M} \sum_{t=T-M+1}^{T} \mathbf{1}_{b_{i,t}(1+\rho) < \wedge \mathbf{b}_{-i,t}},$$

denote firm $i$'s sample demand; i.e., the empirical probability with which $i$ wins an auction at any given bid during the monitoring phase. Fix $\rho \in (0,1)$. For $\mu > 0$ small, define tests $(\tau_i^1)_{i \in N}$ as

$$\tau_i^{\mathsf{missing}} \equiv \mathbf{1}\left(\widehat{D}_i(0, T, M) < (1+\rho)\widehat{D}_i(\rho, T, M) - \mu\right).$$

In words, firm $i$ fails test $\tau_i^{\mathsf{missing}}$ if its sample demand $\widehat{D}_i(\rho, T, M)$ is inelastic at $\rho = 0$. We note that the data in Figure 1 fails tests $(\tau_i^{\mathsf{missing}})$: when the distribution of bid differences

lar, it shows that such patterns may arise when cartels have access to a mediator that helps them coordinate their bids.

$\Delta_{i,t}$ has no mass at $\Delta = 0$, we have $D_i(\rho, T, M) \approx D_i(0, T, M)$ for all $\rho > 0$ small.

In Chassang et al. (2022), we show that testing for missing bids is a safe test.[18]

**Proposition 6** (Chassang et al. (2022)). *Tests* $(\tau_i^{\textsf{missing}})_{i \in N}$ *are asymptotically safe.*

The intuition behind Proposition 6 is simple: when firm $i$'s sample demand is inelastic, it is stage-game profitable for the firm to increase its bids. Hence, such bidding profiles cannot arise when firm $i$ is competitive.

By Proposition 6, adaptive behavior by a cartel to tests $(\tau_i^{\textsf{close}})$ can be detected using safe tests $(\tau_i^{\textsf{missing}})$. Importantly, asymptotically safe tests have the attractive property that they be freely combined.

**Remark 1.** *If* $\tau_i$ *and* $\tau_i'$ *are asymptotically safe tests, then* $\min\{\tau_i, \tau_i'\}$ *and* $\max\{\tau_i, \tau_i'\}$ *are also asymptotically safe tests.*

Remark 1 implies that a regulator may screen for collusion using a combination of asymptotically safe tests without inadvertently enhancing firms' ability to collude.

# 7 Conclusion

We propose an equilibrium model of data-driven screening for cartel behavior in which cartel members can adapt their bidding behavior to undermine regulatory oversight. We emphasize the value of safe tests designed to fail firms whose behavior cannot be explained by any competitive model. We show that such tests cannot help cartels sustain new collusive equilibria, and that they can be freely combined to create new safe tests.

We identify a safe test of interest, which identifies whether a positive fraction of winning bids are very close to the second lowest bid. Such tests rule out optimal collusive behavior in plausible settings, and may strictly reduce the set of enforceable equilibrium values. This

[18] Importantly, the test is safe even if the regulator places no restrictions on the set of economic environments.

test is related to screens used by regulators in practice, and data from procurement auctions in Japan provides suggestive evidence that cartel members do in fact adapt to this test. We show that the bidding patterns generated by the cartels' adaptive response to these tests can be exploited to generate other safe tests.

# Appendix

# A Proofs

**Proof of Proposition 1.** The result follows from two observations. First, for every strategy profile $\overline{\sigma}_{-i}$ of $i$'s opponents and every history $h_{i,t}$ with $t \leq T - M$, firm $i$ can guarantee itself a payoff of at least $0 - G(M)\delta^{T-M}K_T$ by playing a static best-response to $\overline{\sigma}_{-i}$ at all periods $s \geq t$. Second, since firms' flow profits are bounded above by $r = 1$, we have that $W^E(\overline{\sigma}, h_{i,t}) \leq 1$ for all $i$, all $\overline{\sigma}$ and all $h_{i,t}$. Hence, for any $E \in \mathcal{E}$ and any $\sigma \in \Sigma_{T,M}(E)$, firms' payoffs (including expected penalties) at $t < T - M$ must lie $\mathcal{B}_E^{T-M-t}([-G(M)\delta^{T-M}K_T, 1]^n)$.
■

**Proof of Proposition 2.** Fix an environment $E \in \mathcal{E}$. Recall that for every strategy profile $\overline{\sigma}$, every firm $i \in N$ and every history $h_{i,t}$ of firm $i$, $V_i^E(\overline{\sigma}, h_{i,t})$ denotes firm $i$'s expected continuation payoff (excluding penalties) at history $h_{i,t}$ under $\overline{\sigma}$, and

$$P_i^E(\overline{\sigma}, h_{i,t}) = \mathbf{1}_{t \leq T}\mathbb{E}_{E,\overline{\sigma}}\left[\tau_i \delta^{T-t} K_T | h_{i,t}\right]$$

denotes firm $i$'s expected discounted penalty at history $h_{i,t}$ under $\overline{\sigma}$. Firm $i$'s total expected payoff at history $h_{i,t}$ under $\overline{\sigma}$ is $W^E(\overline{\sigma}, h_{i,t}) = V_i^E(\overline{\sigma}, h_{i,t}) - P_i^E(\overline{\sigma}, h_{i,t})$.

Fix $T > 0$ and $\sigma = (\sigma_i)_{i \in N} \in \Sigma_{T,M}(E)$. Pick $\epsilon > 0$ and let $\widehat{T}$ be such that $\delta^{\widehat{T}}(1 + \overline{c}) < \frac{\epsilon}{2}$, where $\overline{c}$ is an upper bound to firms' costs (and so firms' flow payoffs from the auction are bounded above by $r = 1$ and bounded below by $-\bar{c}$). Pick a history $h_{i,t}$, and a strategy $\tilde{\sigma}_i \neq \sigma_i$ for player $i$. Let $\hat{\sigma}_i(\tilde{\sigma}_i)$ be a strategy that coincides with $\tilde{\sigma}_i$ at all histories $h_{i,s}$ of length $s \leq t + \widehat{T}$, and that plays a static best-response to $\sigma_{-i}$ at all histories $h_{i,s}$ of length $s > t + \widehat{T}$. Note then that

$$|V_i^E((\tilde{\sigma}_i, \sigma_{-i}), h_{i,t}) - V_i^E((\hat{\sigma}_i(\tilde{\sigma}_i)), \sigma_{-i}), h_{i,t})| < \frac{\epsilon}{2}. \tag{1}$$

Since $\sigma \in \Sigma_{T,M}(E)$, it must be that

$$\begin{aligned} & V_i^E(\sigma, h_{i,t}) - P_i^E(\sigma, h_{i,t}) \geq V_i^E((\hat{\sigma}_i(\tilde{\sigma}_i), \sigma_{-i}), h_{i,t}) - P_i^E((\hat{\sigma}_i(\tilde{\sigma}_i), \sigma_{-i}), h_{i,t}) \\ \Longleftrightarrow\ & V_i^E(\sigma, h_{i,t}) - V_i^E((\hat{\sigma}_i(\tilde{\sigma}_i), \sigma_{-i}), h_{i,t}) \geq P_i^E(\sigma, h_{i,t}) - P_i^E((\hat{\sigma}_i(\tilde{\sigma}_i), \sigma_{-i})), h_{i,t}). \end{aligned} \tag{2}$$

Moreover, since tests $(\tau_i^T)$ are asymptotically safe, for $T > t + \widehat{T} + M$ it must be that

$$\begin{aligned} P_i^E(\hat{\sigma}_i(\tilde{\sigma}_i), h_{i,t}) &= \mathbb{E}_{E,\sigma}\left[\tau_i^T \delta^{T-t} K_T | h_{i,t}\right] \\ &\leq \delta^{T-t} K_T G(M) = \delta^{-t} K G(M), \end{aligned} \tag{3}$$

where the last equality uses $K_T = \delta^{-T} K$. Since $\lim_{M \to 0} G(M) = 0$ and since $M \sim O(T^x)$ for some $x \in (0, 1)$, the right-hand side of (3) goes to zero as $T \to \infty$. Let $\overline{T} > 0$ be such that $KG(M) \leq \frac{\epsilon}{2}$ for all $T > \overline{T}$. Then, for all $T > \overline{T}$, we have

$$V_i^E(\sigma, h_{i,t}) \geq V_i^E((\hat{\sigma}_i(\tilde{\sigma}), \sigma_{-i}), h_{i,t}) - \delta^{-t}\frac{\epsilon}{2} \geq V_i^E((\tilde{\sigma}_i, \sigma_{-i}), h_{i,t}) - \delta^{-t}\epsilon, \tag{4}$$

where the first inequality uses (2), (3) and $P_i(\sigma, h_{i,t}) \geq 0$, and the second inequality uses (1) and $\delta^{-t} \geq 1$.

We now use (4) to establish the result. Towards a contradiction, suppose that $\Sigma_\infty(E) \not\subset \Sigma(E)$. Hence, there exists $\sigma^* \notin \Sigma(E)$, and sequences $(T^k) \to \infty$, $(\sigma^k) \to \sigma^*$ with $\sigma^k \in \Sigma_{T^k, M^k}(E)$ for all $k$.

Since $\sigma^* \notin \Sigma(E)$, there exists $i \in N$, a history $h_{i,t}$, a deviation $\tilde{\sigma}_i \neq \sigma_i^*$ and a scalar $\eta > 0$ such that

$$V_i^E((\tilde{\sigma}_i, \sigma_{-i}^*), h_{i,t}) > V_i^E(\sigma^*, h_{i,t}) + \eta. \tag{5}$$

Pick $\epsilon > 0$ such that $\epsilon\delta^{-t} < \frac{\eta}{4}$. By our arguments above (see equation (4)), for all $k$ large enough it must be that

$$V_i^E(\sigma^k, h_{i,t}) \geq V_i^E((\tilde{\sigma}_i, \sigma_{-i}^k), h_{i,t}) - \delta^{-t}\epsilon > V_i^E((\tilde{\sigma}_i, \sigma_{-i}^k), h_{i,t}) - \frac{\eta}{4}. \tag{6}$$

Since payoffs are continuous in strategies, and since $\sigma^k \to \sigma^*$, for all $k$ large enough it must be that

$$|V_i^E(\sigma^k, h_{i,t}) - V_i^E(\sigma^*, h_{i,t})| \leq \frac{\eta}{4} \tag{7}$$

$$|V_i^E((\tilde{\sigma}_i, \sigma_{-i}^k), h_{i,t}) - V_i^E((\tilde{\sigma}_i, \sigma_{-i}^*), h_{i,t})| \leq \frac{\eta}{4} \tag{8}$$

Combining (6) with (7) and (8) we get

$$V_i^E(\sigma^*, h_{i,t}) - V_i^E((\tilde{\sigma}_i, \sigma_{-i}^*, h_{i,t}) \geq -\frac{3}{4}\eta,$$

which contradicts (5). Hence, it must be that $\Sigma_\infty(E) \subset \Sigma(E)$. Since this holds for all $E \in \mathcal{E}$, this completes the proof. ■

**Proof of Corollary 1.** Fix $E \in \mathcal{E}$, $\sigma \in \Sigma_{T,M}(E)$, $i \in N$ and a history $h_{i,T-M}$. Firm $i$'s expected payoff at $h_{i,T-M}$ under $\sigma, E$ is $V_i^E(\sigma, h_{i,T-M}) - P_i^E(\sigma, h_{i,T-M})$. Let $\hat{\sigma}_i$ be a strategy under which firm $i$ plays a static best-response to $\sigma_{-i}$ at all histories. Firm $i$'s payoff from playing according to $\hat{\sigma}_i$ when her opponents play according to $\sigma_{-i}$ at history $h_{T-M}$ satisfies

$$V_i^E((\hat{\sigma}_i, \sigma_{-i}), h_{i,T-M}) - P_i^E((\hat{\sigma}_i, \sigma_{-i}), h_{i,T-M}) \geq 0 - G(M)\delta^M K_T = -G(M)\delta^{-(T-M)}K, \tag{9}$$

where the inequality follows since $\tau_i$ is an asymptotically safe test, and since firm $i$'s static best response each period must give $i$ a payoff weakly larger than 0 and the equality uses $K_T = \delta^{-T} K$. Since $\sigma$ is an equilibrium, and since $V_i^E(\sigma, h_{i,T-M}) \leq r = 1$, we have

$$\begin{aligned}
1 - P_i^E(\sigma, h_{i,T-M}) &\geq V_i^E(\sigma, h_{i,T-M}) - P_i^E(\sigma, h_{i,T-M}) \\
&\geq V_i^E((\hat{\sigma}_i, \sigma_{-i}), h_{i,T-M}) - P_i^E((\hat{\sigma}_i, \sigma_{-i}), h_{i,T-M}) \geq -G(M)\delta^{-(T-M)}K \\
&\Longrightarrow \quad P_i^E(\sigma, h_{i,T-M}) \leq 1 + G(M)\delta^{-(T-M)}K.
\end{aligned}$$

Using $P_i^E(\sigma, h_{i,T-M}) = \delta^M K_T \mathbb{E}_{E,\sigma}\left[\tau_i | h_{i,T-M}\right] = \delta^{-(T-M)} K \mathbb{E}_{E,\sigma}\left[\tau_i | h_{i,T-M}\right]$, we get

$$\mathbb{E}_{E,\sigma}\left[\tau_i | h_{i,T-M}\right] \leq \frac{1}{\delta^{-(T-M)}K} + G(M).$$

Hence, if $M \sim O(T^x)$ for some $x \in (0,1)$, $\mathbb{E}_{E,\sigma}\left[\tau_i | h_{i,T-M}\right] = \mathsf{prob}_{E,\sigma}[\tau_i = 1 | h_{i,T-M}]$ converges to 0 as $T \to \infty$. This completes the proof. $\blacksquare$

**Proof of Proposition 4.** Fix $E \in \mathcal{E}$, and let $\sigma$ be a strategy profile under which firm $i$ is competitive during the monitoring phase. Fix an on-path history $h_{i,t}$ with $t \in [T-M+1, T]$ such that $\sigma_i(h_{i,t}) = b_{i,t} \in [0,1]$.

For each $b \in [0,1]$, define $D_i(b|h_{i,t}) \equiv \mathsf{prob}_{E,\sigma}(\wedge \mathbf{b}_{-i,t} \succ b | h_{i,t})$, where $\wedge \mathbf{b}_{-i,t} \succ b$ denotes the event that either $\wedge \mathbf{b}_{-i,t} > b$, or $\wedge \mathbf{b}_{-i,t} = b$ but ties are broken in favor of bidder $i$. Hence, $D_i(b|h_{i,t})$ is the probability with which firm $i$ expects to win the auction if she places bid $b$ at history $h_{i,t}$, given strategy profile $\sigma$.

Since firm $i$ is competitive during the monitoring phase under $\sigma$, for any $\Delta > 0$ it must be that

$$D_i(b_{i,t} - \Delta | h_{i,t})(b_{i,t} - \Delta - c_{i,t}) \leq D_i(b_{i,t} | h_{i,t})(b_{i,t} - c_{i,t}).$$

Moreover, since bidding is costly, it must be that $D_i(b_{i,t}|h_{i,t})(b_{i,t} - c_{i,t}) \geq \hat{\kappa} > 0$, and so

$b_{i,t} - c_{i,t} \geq \hat{\kappa}$. Combining this with the inequality above, we get that for all $\Delta > 0$ small,

$$\begin{aligned} D_i(b_{i,t} - \Delta | h_{i,t}) \leq D_i(b_{i,t}|h_{i,t}) \frac{b_{i,t} - c_{i,t}}{b_{i,t} - c_{i,t} - \Delta} \leq D_i(b_{i,t}|h_{i,t}) \frac{\hat{\kappa}}{\hat{\kappa} - \Delta} \\ \implies D_i(b_{i,t} - \Delta | h_{i,t}) - D_i(b_{i,t}|h_{i,t}) \leq \frac{\Delta}{\hat{\kappa}} D_i(b_{i,t} - \Delta | h_{i,t}) \leq \frac{\Delta}{\hat{\kappa}}. \end{aligned} \tag{10}$$

Note that inequality (10) implies $\lim_{\Delta \searrow 0} D_i(b_{i,t} - \Delta | h_{i,t}) = D_i(b_{i,t}|h_{i,t})$; i.e., at history $h_{i,t}$, the distribution of $\wedge \mathbf{b}_{-i,t}$ cannot have a mass point at $b_{i,t}$. Hence, $D_i(b_{i,t}|h_{i,t}) = \mathsf{prob}_{E,\sigma}(\wedge \mathbf{b}_{-i,t} > b_{i,t}|h_{i,t})$, and so (10) implies that, for all $\rho > \frac{\Delta}{\hat{\kappa}}$,

$$\mathsf{prob}_{E,\sigma}(\wedge \mathbf{b}_{-i,t} \in (b_{i,t} - \Delta, b_{i,t}] | h_{i,t}) \leq D_i(b_{i,t} - \Delta | h_{i,t}) - D_i(b_{i,t}|h_{i,t}) \leq \frac{\Delta}{\hat{\kappa}} < \rho. \tag{11}$$

We now use (11) to show that tests $(\tau_i^{\mathsf{close}})$ are asymptotically safe when $\rho > \frac{\Delta}{\hat{\kappa}}$. For any $t$, define

$$\varepsilon_t \equiv \mathbf{1}_{\wedge \mathbf{b}_{-i,t} \in (b_{i,t} - \Delta, b_{i,t}]} - \mathsf{prob}_{E,\sigma}(\wedge \mathbf{b}_{-i,t} \in (b_{i,t} - \Delta, b_{i,t}] | h_{i,t}).$$

For all $t \in [T - M + 1, T]$, let $S_{t,M} = \sum_{s=T-M+1}^{t} \varepsilon_s$. Note that, for all $s$, $\mathbb{E}_{E,\sigma}[\varepsilon_s | h_{i,s-1}] = 0$. Hence, $S_{t,M} = \sum_{s=T-M+1}^{t} \varepsilon_s$ is a Martingale with respect to $(h_{i,t})$, with the absolute value of its increments bounded above by 1. By the Azuma-Hoeffding inequality, for any $\alpha > 0$

$$\mathsf{prob}_{E,\sigma}(S_{T,M} > \alpha M) \leq \exp\left(-\frac{\alpha^2 M}{2}\right). \tag{12}$$

The probability that firm $i$ fails test $\tau_i^{\mathsf{close}}$ under $E, \sigma$ is given by

$$\begin{aligned} \mathsf{prob}_{E,\sigma}(\tau_i^{\mathsf{close}} = 1) = & \mathsf{prob}_{E,\sigma}\left(\frac{1}{M} \sum_{t=T-M+1}^{T} \mathbf{1}_{\wedge \mathbf{b}_{-i,t} \in (b_{i,t} - \Delta, b_{i,t}]} > \rho\right) \\ = & \mathsf{prob}_{E,\sigma}\left(\frac{1}{M} S_{T,M} > \rho - \frac{1}{M} \sum_{t=T-M+1}^{T} \mathsf{prob}_{E,\sigma}(\wedge \mathbf{b}_{-i,t} \in (b_{i,t} - \Delta, b_{i,t}] | h_{i,t})\right) \\ \leq & \mathsf{prob}_{E,\sigma}\left(\frac{1}{M} S_{T,M} > \rho - \frac{\Delta}{\hat{\kappa}}\right), \end{aligned} \tag{13}$$

where the last inequality follows from (11). Combining (13) with (12), and letting $\alpha = \rho - \frac{\Delta}{\hat{\kappa}} > 0$, we get that

$$\begin{aligned}\mathsf{prob}_{E,\sigma}(\tau_i^{\mathsf{close}} = 1) =& \mathsf{prob}_{E,\sigma}\left(\frac{1}{M}\sum_{t=T-M}^{T} \mathbf{1}_{\wedge \mathbf{b}_{-i,t} \in (b_{i,t}-\Delta, b_{i,t}]} > \rho\right) \\ \leq& \mathsf{prob}_{E,\sigma}(S_{T,M} > \alpha M) \leq \exp\left(-\frac{\alpha^2 M}{2}\right)\end{aligned}$$

This completes the proof. ■

**Proof of Proposition 5.** Consider first part (i). Since the auction runs only if both firms participate, the cartel's flow payoff in the repeated game is bounded above by $(1-\delta)(r-2\kappa) = (1-\delta)(1-2\kappa)$. Consider the following strategy $\sigma_i^{\mathsf{coll}}$:

- at the initial history $h_0$, or at any history $h_t$ with $b_{j,s} = r$ for $j = 1, 2$, $s < t$, bid $b_{i,t} = r$;
- at any other history $h_t$, play the static Nash equilibrium.[19]

One can verify that, when $\delta \geq \underline{\delta} = \frac{1}{1-\kappa}\frac{1}{2}$, both firms playing according to $\sigma_i^{\mathsf{coll}}$ is an equilibrium of the repeated game without a regulator. Hence, $\overline{V}^{\mathsf{bmk}} = 1 - 2\kappa$.

We now turn to part (ii). We start by providing an upper bound to the cartel's payoffs during the monitoring phase. Let $b_{(1),s}$ denote the winning bid at period $s$, and for any $t \in [T-M+1, T+1]$ let $B_{t-1} = \sum_{s=T-M+1}^{t-1} b_{(1),s}$ denote the cumulative sum of winning bids during the testing phase prior to $t$ (with $B_{T-M} = 0$). Note that, if both firms participate at all periods $t = T-M+1, ..., T$, at least one firm $i \in \{1, 2\}$ fails test $\tau_i^{\mathsf{close}}$ if $\frac{1}{M}B_T > \overline{B} \equiv 1 - (1-2\rho)\Delta$. Define $\widehat{W} \equiv \overline{B} - 2\kappa$.

For each history $h_t = (\mathbf{b}_s)_{s<t}$ with $t \in [T-M+1, T+1]$ define state $z_t = (t, B_{t-1})$ and let

$$\Phi(z_{T+1}) = 1 - 2\kappa - \delta^{-1} K_T \mathbf{1}_{B_T > M \times \overline{B}}.$$

[19]With $\kappa > 0$, the stage game admits a Nash equilibrium in which both firms always stay out of the auction.

Note that $\delta\Phi(z_{T+1})$ is an upper bound on the sum of firms' equilibrium payoffs at the end of period $T$, after bidding is done but before any penalties are paid.

For each $z_t = (t, B_{t-1})$ with $t \in [T+1-M, T]$, define

$$\Phi(z_t) = \sup_{b_t \sim F \in \Delta([0,1])} \mathbb{E}_F[(1-\delta)(b_t - 2\kappa) + \delta\Phi(t+1, B_{t-1}+b_t)] \text{ s.t.} \tag{14}$$
$$\forall b_t \in \mathsf{supp}\, F, \quad b_t \leq \frac{\delta}{1-\delta}\Phi(t+1, B_{t-1}+b_t).$$

We note the following two observations. First, (14) always admits a deterministic solution (i.e., a solution such that winning bid $b_t$ is deterministic for all $z_t$). Indeed, (14) is a deterministic dynamic programing problem, and hence admits a deterministic solution. Second, whenever penalty $K_T$ is sufficiently large (e.g., larger than $\overline{K} \equiv 1 - 2\kappa$), the solution to (14) will be such that firms pass the test with probability 1: i.e., $B_T \leq M\overline{B}$. From now on, we assume $K_T \geq \overline{K}$. For each $t = T+1-M, ..., T+1$, let $\overline{\Phi}_t$ be the value of Program (14) at $t$. Since $z_{T+1-M} = (T+1-M, 0)$ regardless of the bidding history, the solution to (14) does not depend on bidding behavior prior to $T+1-M$.

We now show that, for any equilibrium $\sigma$ and any history $h_{T+1-M}$ of length $T+1-M$, the cartel's payoff under $\sigma$ at history $h_{T+1-M}$ is bounded above by $\overline{\Phi}_{T+1-M}$. To establish this result, we show that for any period $t$ during the testing phase with history $h_t = (\mathbf{b}_s)_{s<t}$ and with state $z_t = (t, B_{t-1})$, if history $h_t$ is such that no firm $i \in \{1,2\}$ would fail the test if it were to not participate at any auction $s \geq t$, then the cartel's equilibrium payoff at $h_t$ under $\sigma$ is bounded above by $\Phi(z_t)$. Note that this would immediately imply that $\overline{\Phi}_{T+M-1}$ is an upper bound to equilibrium payoffs at any history $h_{T+1-M}$.[20]

Note first that, since $\Phi(z_{T+1}) = 1-2\kappa$ for any history $h_{T+1}$ such that both firms pass the test, cartel equilibrium payoffs at $h_{T+1}$ are indeed bounded above by $\Phi(z_{T+1})$.[21] Towards an induction, suppose that the result holds for all histories $h_{\hat{s}}$ of length $\hat{s} = t+1, ..., T+1$

[20]Indeed, all histories $h_{T+1-M}$ of length $T+1-M$ have $B_{T-M} = 0$ and have the property that both firms would pass test $\tau_i^{\mathsf{close}}$ if they didn't participate at any auction $s \geq T+1-M$.

[21]Indeed, if $h_{T+1}$ is such that both firms pass the test, then $B_T \leq M \times \overline{B}$.

with the property that both firms would pass the test if they stopped participating. Fix an equilibrium $\sigma$ and a history $h_t = (\mathbf{b}_s)_{s<t}$ of length $t$, with $t$ during the monitoring phase, with the property that both firms would pass the test if they stopped participating. Let $b_{(1),t}$ denote the equilibrium winning bid at $h_t$. Note that, if both firms participate at $h_t$, then for $i = 1, 2$ we must have

$$(1-\delta)(x_{i,t}b_{(1),t} - \kappa) + \delta W_{i,t+1} \geq (1-\delta)(b_t - \kappa), \tag{15}$$

where $x_{i,t}$ denotes the probability with which $i$ wins the auction at time $t$ and $W_{i,t+1}$ denotes $i$'s continuation value given history $h_{t+1} = h_t \sqcup \mathbf{b}_t$ (including expected penalties).[22] Summing across both players, and using $x_{1,t} + x_{2,t} \leq 1$ and $W_{1,t+1} + W_{2,t+1} = W_{t+1}$, we get

$$\begin{aligned}(1-\delta)(b_{(1),t} - 2\kappa) + \delta W_{t+1} &\geq 2(1-\delta)(b_{(1),t} - \kappa)\\ \iff \frac{\delta}{1-\delta} W_{t+1} &\geq b_{(1),t}.\end{aligned}$$

If bids $\mathbf{b}_t = (b_{1,t}, b_{2,t})$ are such that firm $i \in \{1, 2\}$ fails the test under $h_t \sqcup \mathbf{b}_t$ if it were to stop participating, then we have $W_t \leq \Phi(z_t)$ whenever $K \geq \overline{K}$. If not, then by the induction hypothesis we have that $W_{t+1} \leq \Phi(z_{t+1})$; and so $W_t \leq \Phi(z_t)$. Hence, equilibrium payoffs at any history $h_{T-M+1}$ are bounded above by $\overline{\Phi}_{T-M+1}$.

We now show that, for all $\epsilon > 0$, $\overline{\Phi}_t \leq \widehat{W} + \epsilon = \overline{B} - 2\kappa + \epsilon$ for some $t \in [T - M + 1, T]$ whenever $M$ is large enough. Since the solution to (14) is deterministic, we have that for all $t = T - M + 1, ..., T$, $\overline{\Phi}_t = (1-\delta)(b_t - 2\kappa) + \delta\overline{\Phi}_{t+1}$, or

$$b_t = \frac{1}{1-\delta}\left(\overline{\Phi}_t - \delta\overline{\Phi}_{t+1}\right) + 2\kappa.$$

[22] To see why (15) must hold under $\sigma$, recall that $h_t$ is such that $i$ would not fail the test if she were to not participate any longer. Hence, (15) must hold since firm $i$ can obtain a payoff equal to the right-hand side by undercutting bid $b_{(1),t}$ at $t$ and not participating at any future date.

Summing over periods $t = T - M + 1, ..., T$ and dividing by $M$ we get

$$\frac{1}{M}\sum_{t=T-M+1}^{T} b_t = \frac{1}{M}B_T = \frac{1}{M}\left(\sum_{t=T-M+2}^{T}\overline{\Phi}_t + \frac{1}{1-\delta}\left(\overline{\Phi}_{T-M+1} - \delta\overline{\Phi}_T\right)\right) + 2\kappa. \tag{16}$$

Since firms pass the test under the solution to (14), we have $\frac{1}{M}B_T \leq \overline{B} = \widehat{W} + 2\kappa$, and so (16) gives us

$$\frac{1}{M-1}\sum_{t=T-M+2}^{T}\overline{\Phi}_t \leq \frac{M}{M-1}\widehat{W} + \frac{1}{1-\delta}\frac{1}{M-1}\left(\delta\overline{\Phi}_T - \overline{\Phi}_{T-M+1}\right)$$

Since $\overline{\Phi}_t \leq 1 - 2\kappa$ for all $t$, we have that for all $\epsilon > 0$ there exists $\overline{M}_\epsilon$ such that, for all $M \geq \overline{M}_\epsilon$,

$$\frac{1}{M-1}\sum_{t=T-M+2}^{T}\overline{\Phi}_t \leq \widehat{W} + \epsilon. \tag{17}$$

Equation (17) implies that there exists $t \in [T - M + 2, T]$ such that $\overline{\Phi}_t \leq \widehat{W} + \epsilon$.

Pick $\epsilon > 0$ such that $\widehat{W} + \epsilon < 1 - 2\kappa$. Note that such an $\epsilon > 0$ exists since $\overline{B} \equiv 1 - (1-2\rho)\Delta$ and $\widehat{W} \equiv \overline{B} - 2\kappa$. Assume $M \geq \overline{M} \equiv \overline{M}_\epsilon$. Note then that there exists $\mu > 0$ and $\eta > 0$ such that, for all $V \leq \widehat{W} + \epsilon$, $\frac{\delta}{1-\delta}V \leq V + 2\kappa - \eta$ for all $\delta \in [\underline{\delta}, \underline{\delta} + \mu]$.[23] We now show that, for all $\delta \in [\underline{\delta}, \underline{\delta} + \mu]$, $\overline{\Phi}_{T-M+1} \leq \widehat{W} + \epsilon$. From our arguments above, we know that there exists $t \in [T - M + 2, T]$ such that $\overline{\Phi}_t \leq \widehat{W} + \epsilon$. Note then that

$$\begin{aligned}
\overline{\Phi}_{t-1} &= (1-\delta)b_{t-1} + \delta\overline{\Phi}_t \\
&\leq (1-\delta)\left(\frac{\delta}{1-\delta}\overline{\Phi}_t - 2\kappa\right) + \delta\overline{\Phi}_t \\
&\leq \widehat{W} + \epsilon - (1-\delta)\eta,
\end{aligned}$$

[23]To see why, note that, for all $V \leq \widehat{W} + \epsilon$, we have

$$V + 2\kappa - \frac{\delta}{1-\underline{\delta}}V = 2\kappa\left(1 - \frac{V}{1-2\kappa}\right) > 0,$$

where the equality follows since $\frac{\delta}{1-\underline{\delta}} = \frac{1}{1-2\kappa}$ and the strict inequality follows since $V \leq \widehat{W} + \epsilon < 1 - 2\kappa$.

where the first inequality uses $b_{t-1} \leq \frac{\delta}{1-\delta}\overline{\Phi}_t$, and the second inequality uses $\overline{\Phi}_t \leq \widehat{W} + \epsilon$ and $\frac{\delta}{1-\delta}\overline{\Phi}_t \leq \overline{\Phi}_t + 2\kappa - \eta$. If $t-1 = T-M+1$, we are done. Otherwise, we can repeat the same argument until we obtain $\overline{\Phi}_{T-M+1} < \widehat{W} + \epsilon$.

Finally, we show that for all $\varepsilon > 0$ and all $\delta \in [\underline{\delta}, \underline{\delta}+\mu]$, there exists $\overline{T}$ such that $\overline{V}^{\mathsf{bmk}}_{T,M} < \varepsilon$ for all $M > \overline{M}$, $T - M > \overline{T}$.

Fix $\sigma \in \Sigma_{T,M}(E)$ and a history $h_t$ with $t \leq T - M$. Let $b_{(1),t}$ denote the winning bid at $h_t$. Note that, if both firms participate, then for $i = 1, 2$ we must have

$$(1-\delta)(x_{i,t}b_{(1),t} - \kappa) + \delta W_{i,t+1} \geq (1-\delta)(b_{(1),t} - \kappa),$$

where $x_{i,t}$ denotes the probability with which $i$ wins the auction at time $t$ and $W_{i,t+1}$ denotes $i$'s continuation value given history $h_{t+1} = h_t \sqcup \mathbf{b}_t$ (including possible penalties). Summing across both players, and using $x_{1,t} + x_{2,t} \leq 1$ and $W_{1,t+1} + W_{2,t+1} = W_{t+1}$, we get

$$\begin{aligned}(1-\delta)(b_{(1),t} - 2\kappa) + \delta W_{t+1} &\geq 2(1-\delta)(b_{(1),t} - \kappa) \\ \iff \frac{\delta}{1-\delta}W_{t+1} &\geq b_{(1),t}.\end{aligned}$$

For each value $W \in \mathbb{R}_+$, define operator $\Psi : \mathbb{R}_+ \to \mathbb{R}_+$ as

$$\Psi(W) \equiv (1-\delta)\left(\max\left\{\min\left\{1, \frac{\delta}{1-\delta}W\right\} - 2\kappa, 0\right\}\right) + \delta W.$$

Then, for any history $h_t$ with $t \leq T - M$, $\Psi(W)$ is an upper bound to the cartel's payoff at a history $h_t$ whenever the cartel's continuation value at $t+1$ is bounded above by $W$. Since by our arguments above cartel's continuation payoff at any history $h_{T-M+1}$ is bounded above by $\overline{\Phi}_{T-M+1} \leq \widehat{W} + \epsilon$, we have that $\overline{V}^{\mathsf{bmk}}_{T,M} \leq \Psi^{T-M}(\widehat{W} + \epsilon)$. Recall that $\frac{\delta}{1-\delta}W \leq W + 2\kappa - \eta$ for all $W \leq \widehat{W} + \epsilon$. Hence, for all $W \leq \widehat{W} + \epsilon$ we have $\Psi(W) \leq \max\{W - (1-\delta)\eta, \delta W\}$. And so, for $T - M$ sufficiently large, $\varepsilon > \Psi^{T-M}(\widehat{W} + \epsilon) \geq \overline{V}^{\mathsf{bmk}}_{T,M}$. This completes the proof.

■

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