Revenue guarantee in auction with common prior∗

Takuro Yamashita†

October 26, 2015

Abstract

This paper considers auction environments with a (possibly correlated) common prior over bidders’ values, where each bidder may have additional information (e.g., through information acquisition). We characterize the optimal mechanisms in terms of the expected revenue that is guaranteed given whatever additional information is available to the bidders. Specifically, we show that (i) a second-price auction is optimal among all the efficient mechanisms, (ii) it is “approximately” optimal among all the mechanisms, and (iii) a sequential posted-price mechanism is optimal in certain dynamic environments.

1 Introduction

In many real auction, bidders’ valuations for auctioned objects exhibit correlation.1 Despite its practical importance of optimal auction mechanism

∗I am extremely indebted to Gabriel Carroll for his helpful discussion and comments. I thank Tilman Börgers, Antonio Penta, and Benjamin Brooks, Stephen Morris, seminar participants at Toulouse School of Economics, Hitotsubashi University, SAET 2014, and Aalto University for their comments and suggestions.

†Toulouse School of Economics, takuro.yamashita@tse-fr.eu

1For example, imagine auctions of oil tracts, spectra, sovereign bonds, and so on.

1
design with correlated private information, however, most of the papers in the Bayesian mechanism design literature focus on independently distributed valuations, and not much has been known about the correlated cases. This may be partly because of the extremely positive result obtained by Crémé and McLean (1985, 1988) in “generic” correlated environments.\(^2\) Crémé and McLean (1985, 1988) show that, with a generic correlated distribution, \textit{any} allocation rule is implementable without \textit{any} information rent, and in this sense, the first-best outcome for the seller is always possible, even though the valuations are the bidders’ private information. This observation is very different from what is known in the independent case (e.g., Myerson (1981)), where implementable allocation rules must be monotonic and bidders earn some information rent.

This stark difference or discontinuity between the independent and correlated cases has been considered to be puzzling in the literature.\(^3\) On one hand, it seems that the bidders’ valuations are correlated in many auctions. On the other hand, however, the observations in independent cases may seem to be more sensible, in that the bidders’ private information restricts implementable objectives and the seller’s ability to extract information rent. Also, the optimal mechanism obtained by Crémé and McLean (1985, 1988), which may be interpreted as a combination of a second-price auction and side bets, is often criticized as highly unrealistic.

A crucial (implicit) assumption in Crémé and McLean (1985, 1988) is that the bidders cannot have additional information about each other’s valuation (in addition to the correlated common prior). For example, this means that each bidder cannot engage in information acquisition about other bidders’ valuations. This assumption is crucial for their side-bet mechanism to extract information rents, where each bidder “bets” on other bidders’ valuations. Naturally, if he can acquire additional information about other

\(^2\)See also McAfee and Reny (1992), Heifetz and Neeman (2006), Chen and Xiong (2013), and Gizatulina and Hellwig (2014).

\(^3\)See, for example, Crémé and McLean (1985, 1988), McAfee and Reny (1992), and Milgrom (2004).
bidders, he would have a rather strong incentive to do so, because then he could earn positive (possibly large) information rent. This means, in turn, the seller can no longer extract all the surplus of the bidders.\footnote{This impossibility of full-surplus extraction is formally observed by Bikhchandani (2010).} Also, this assumption of no information acquisition seems unrealistic in many auction environments. For example, in auction of oil tracts, bidders (e.g., oil companies) often have some technologies to acquire more precise information about the tracts before the bidding stage.

This suggests that allowing for possibilities of additional information (e.g., through information acquisition) is important for mechanism design with correlated information. Therefore, in this paper, we assume that each bidder may have arbitrary additional information (about others’ private information), and the seller does not know what kind of additional information is available to each bidder. Given such “uncertainty” or “ambiguity” about additional information, our goal is to characterize the highest expected revenue that can be guaranteed given whatever additional information the bidders might have. Although there may be many other ways to model the bidders’ additional information, such a “pessimistic” approach may be reasonable when bidders have more expertise than the seller (e.g., auction of oil tracts), so that it is difficult for the seller to know what kind of information acquisition technologies are ever available to the bidders. More generally, avoiding any \textit{ad hoc} restriction on the bidders’ possible additional information, we can avoid the optimal mechanism highly depending on the structure of additional information. This feature may be preferable in view of the “detail-freeness” in Wilson (1987).

The paper is structured as follows. In Section 2, we introduce a single-good, private-value auction model with correlated common prior over bidders’ valuations.\footnote{Although some results can be extended to non-auction environments or interdependent-value (or common-value) environments, the main part of the paper focuses on this simple setting to convey clearer intuition.} We formally define the bidders’ (arbitrary) additional in-
formation and the concept of revenue guarantee. In private-value auction, each agent knows his willingness-to-pay for the object, and in this sense, such additional information is payoff-irrelevant. However, it could be important to determine his (possibly high-order) belief about the other bidders’ private information, and hence important for determining his behavior. In particular, the side-bet mechanism of Crémer and McLean (1985, 1988) can no longer guarantee the first-best level of expected revenue with additional information, and moreover, it often fails to be optimal.

Section 3 provides characterization of the (exact or approximate) highest revenue guarantee, in various settings. In Section 3.1, we show that, among all efficient\(^6\) auction mechanisms, a second-price auction is optimal in the sense of revenue guarantee. Note that this result is not driven by a standard revenue-equivalence argument. Because of a correlated prior and additional information, we cannot apply the standard revenue-equivalence theorem in this environment. Therefore, the result provides a rationale for a benevolent principal (e.g., a government selling its asset) to use a second-price auction, a simple and common auction format, regardless of the valuation distribution.

In Section 3.2, we show that, even if arbitrary (possibly inefficient) auction mechanism is allowed, a second-price auction is still approximately optimal. More precisely, the difference between the highest possible revenue guarantee and the expected revenue in a second-price auction goes to zero as the number of bidders increases, at the exponential rate.\(^7\) Chung and Ely (2007) show that a dominant-strategy mechanism is optimal in revenue guarantee among all auction mechanisms if the bidders’ may have arbitrary (high-order) beliefs about each other’s private information. Our result is partly stronger than theirs in the sense that the set of possible beliefs the bidders may possess in our environment is smaller than that of theirs. In particular, in our model,

\(^6\)An auction mechanism is efficient if a highest-value bidder always wins the object, regardless of additional information of the bidders.

\(^7\)Of course, as the number of bidders increases, the expected second-highest value converges to the expected highest value. However, under many distributions, the convergence rate is much slower than the exponential rate.
the bidders’ beliefs are always consistent with the original correlated prior regardless of additional information, while in their model, the bidders may believe very different priors from each other.\footnote{Börgers (2013) raises a concern about revenue-maximizing Bayesian mechanism design with risk-neutral agents and without a common prior.} Such different priors may be difficult to justify in some contexts, such as those where some data about past auctions of similar objects is publicly available.\footnote{On the other hand, their result is stronger than ours in the sense that they obtain exact optimality. At the technical level, there are other differences too. In this sense, our results are complementary to each other.}

Finally, in Section 3.3, we consider a simple dynamic sales problem. The seller sells an indivisible object to one of the buyers (as in auction), but each buyer $i = 1, \ldots, N$ is available only at time $t = i$. Hence, only one buyer is available at each time period, and the seller essentially faces an optimal stopping problem.\footnote{Although this dynamic model with correlated information is not formally studied by Crémer and McLean (1985, 1988), an analogous rent-extraction is possible without any additional information, for any buyer except for buyer 1.} In this context, we show that a sequential posted-price mechanism is optimal in revenue guarantee, where the posted price at time $t$ in general depends on the buyers’ value reports before $t$ (due to statistical updating implied by a correlated prior) and how many buyers are left after $t$ (due to the option value of waiting).

The common qualitative feature of these results is that, even though we consider Bayesian mechanism design with a correlated common prior, the optimal mechanism in revenue guarantee is a dominant-strategy incentive compatible mechanism (a second-price auction or a sequential price posting). With independent common priors, Myerson (1981) shows optimality of a second-price auction (with a reserve price). Our results may be interpreted as its generalization in correlated environments when revenue guarantee is concerned.\footnote{As in Segal (2003), this sort of result has been conjectured in the literature.}

At the methodological level, the idea that agents may know more than the available information to the “outside observer” is studied by Forges (1993)
and Bergemann and Morris (2013) in game theory (i.e., for arbitrarily fixed games). In the sense that essentially no restriction is made in terms of which additional information may be available, the one studied by Bergemann and Morris (2013) is the closest to our approach. Although our revenue-guarantee problem can be seen as a mechanism-design application of their concept of Bayes correlated equilibrium, we develop a different methodology to analyze our mechanism design problem. We discuss why such a different methodology is necessary in mechanism design in Section 3.

2 Auction environment

There are $N$ bidders, $i = 1, \ldots, N$. Each $i$ knows his value $v_i \in V_i \subseteq [0,1]$ for the object (“private values”), and his utility is given by $v_i q_i - p_i$, where $q_i \in [0,1]$ is the probability that he is assigned the object, and $p_i \in \mathbb{R}$ is his payment. Let $V = \prod_i V_i$. The set of feasible allocations is denoted by $X = \{(q, p) = (q_i, p_i)_{i=1}^N | \sum q_i \leq 1\}$. Given $v = (v_i)_{i=1}^N$, let $v^{(k)}$ denote the $k$-th highest value among $v_1, \ldots, v_N$.

As in the standard Bayesian mechanism design approach, we assume that the bidders commonly know a probability distribution over the values, denoted by $F \in \Delta(V)$, and $F$ is known to the designer as well. The assumption that the distribution over the values being common knowledge may be considered to be a reasonable assumption in some cases, for example, when “similar” goods have been auctioned many times and the data about the value distributions is publicly available.

As opposed to the standard approach, however, we allow for a possibility that each bidder may know more information about the environment, which is unknown to the designer. First, let $S = \prod_i S_i$, each $S_i$ is a measurable space, and $G \in \Delta(V \times S)$. Each $S_i$ is interpreted as the set of additional information available to bidder $i$, and $G$ is a joint probability distribution over $V \times S$. We assume that each $i$ observes both $v_i \in V_i$ and $s_i \in S_i$ before playing a mechanism, and $G$ is commonly known among the bidders.
Definition 1. \((S, G)\) is \(F\)-feasible if the consistency between \(F\) and \(G\) is maintained, i.e., for each measurable \(\tilde{V} \in V\), we have \(G(\tilde{V} \times S) = F(\tilde{V})\).

Let \(G_i(v_i, s_i) \in \Delta(V_{-i} \times S_{-i})\) denote \(i\)'s conditional probability distribution over \(V_{-i} \times S_{-i}\) given his own signal \(v_i, s_i\).

As is clear from the private-value assumption, \(s_i\) is payoff-irrelevant for \(i\), but may play an important role in determining his behavior, because \(s_i\) may be correlated with \(v_{j}\) (and \(s_j\)) for \(j \neq i\).

An auction mechanism is denoted by \(\Gamma = \langle M, q, p \rangle\), where each \(M_i\) is a message set for each \(i\), \(M = \prod_i M_i\), and \((q, p) : M \to X\) is an outcome function. We assume that each \(M_i\) has a message that corresponds to "opting-out", and whenever \(i\) chooses that message, he is assigned \((q_i, p_i) = (0, 0)\).

Given \((V, F; S, G)\) such that \((S, G)\) is \(F\)-feasible, the bidders play a Bayesian equilibrium in mechanism \(\Gamma\). Let \(\sigma_i : V_i \times S_i \to M_i\) be \(i\)'s (pure) strategy in \(\Gamma\). We say that \(\sigma^* = (\sigma^*_i)_{i=1}^N\) is a Bayesian equilibrium if, for each \(i, v_i, s_i, m_i\),

\[
\int_{V_{-i} \times S_{-i}} v_i q_i(\sigma_i^*(v_i, s_i), \sigma_{-i}^*(v_{-i}, s_{-i})) - p_i(\sigma_i^*(v_i, s_i), \sigma_{-i}^*(v_{-i}, s_{-i})) \, dG_i(v_i, s_i) \\
\geq \int_{V_{-i} \times S_{-i}} v_i q_i(m_i, \sigma_{-i}^*(v_{-i}, s_{-i})) - p_i(m_i, \sigma_{-i}^*(v_{-i}, s_{-i})) \, dG_i(v_i, s_i).
\]

Our goal is to study desirable mechanisms in a situation where the designer knows little about this additional payoff-irrelevant information structure. Specifically, we assume that the designer does not know \(S\) and \(G\), except that it is \(F\)-feasible (recall that the designer knows \(F\)), and we evaluate an auction mechanism \(\Gamma\) according to its worst-case expected revenue among all \(F\)-feasible \((S, G)\). In other words, we are interested in the level of expected revenue the designer can guarantee if he assumes that \(F\) is common knowledge and that the bidders are Bayesian (but allows for any additional information structure that is \(F\)-feasible).

Definition 2. A mechanism \(\Gamma\) guarantees expected revenue \(R(\Gamma) \in \mathbb{R}\) if, for any \((S, G)\) that is \(F\)-feasible, there exists a Bayesian equilibrium \(\sigma^*\) such
that

\[
\int_{V \times S} \left[ \sum_i p_i(\sigma^*(v, s)) \right] dG \geq R(\Gamma).
\]

### 3 Revenue guarantee

In the following, we first obtain a lower bound for the highest revenue guarantee by examining dominant-strategy mechanisms. In a dominant-strategy mechanism, truth-telling is a best strategy of every agent regardless of the additional information structure \((S, G)\), and therefore, the expected revenue achieved by a dominant-strategy mechanism with a trivial \((S, G)\) is always achieved with any \((S, G)\).

**Definition 3.** A mechanism \(\Gamma^D = (M, q^D, p^D)\) is a dominant-strategy mechanism if (i) \(M = V\), and (ii) for each \(i, v_i, v'_i \in V_i\) and \(v_{-i} \in V_{-i}\),

\[
v_i q_i(v_i, v_{-i}) - p_i(v_i, v_{-i}) \geq \max\{0, v_i q_i(v'_i, v_{-i}) - p_i(v'_i, v_{-i})\}.
\]

Omitting its message space, we denote a dominant-strategy mechanism simply by \((q^D, p^D)\).

**Lemma 1.** A dominant-strategy mechanism \((q^D, p^D) : V \rightarrow X\) guarantees

\[
\int_v \sum_i p_i^D(v) dF.
\]

Therefore, the expected revenue achieved by the optimal dominant-strategy mechanism provides a lower bound for the highest revenue guarantee among all mechanisms. In general, it is not necessarily the case that this lower bound is tight.

However, in the three applications studied in this paper, we show that this lower bound is either tight or “close to” being tight (in the sense defined later). Here, we explain the basic idea that is common across those applications, although the detail of the proofs varies.
The basic idea is to identify a specific structure of additional information, \((S, G)\), with which the optimal mechanism is (or “is close to”) a dominant-strategy mechanism, even if the designer knows \((S, G)\). If the designer knows \((S, G)\) when he optimally designs a mechanism, the expected revenue in such an optimal mechanism is an upper bound for the highest revenue guarantee. Therefore, if such an optimal mechanism is a dominant-strategy mechanism, then the lower and upper bounds must coincide, implying optimality of the dominant-strategy mechanism in revenue guarantee.

An advantage of this approach is that we can apply the revelation principle. If the designer knows \((S, G)\), then his design problem boils down to a standard Bayesian mechanism design where each agent’s private information is \((v_i, s_i)\), where \((v, s) = (v_i, s_i)_{i\in I}\) follows the distribution \(G\). Of course, if \(G\) exhibits certain correlation, then surplus extraction as in Crémer and McLean (1988) would be possible. In the worst-case scenario across \(F\)-feasible \((S, G)\), as we see in each subsection, \(G\) rather exhibits certain independence condition. More specifically, the agents’ value profile \(v = (v_i)_{i\in I}\) is independently distributed in \(G\) conditional on any realization of payoff-irrelevant signal profile \(s = (s_i)_{i\in I}\). This conditional independence implies optimality of dominant-strategy mechanisms in the three applications below.

In game theory, Forges (1993) and Bergemann and Morris (2013) introduce several versions of incomplete-information correlated equilibria, and at the conceptual level, the revenue-guarantee problem in auction can be seen as a mechanism-design application of the concept of robust prediction in Bergemann and Morris (2013). However, we use different methodology to identify the optimal mechanism. To explain this, recall that, in their robust prediction, Bergemann and Morris (2013) develop the concept of Bayes correlated equilibrium to identify all Bayesian equilibrium outcomes given any \((F\)-feasible) \((S, G)\). Their approach has a great advantage in predicting possible outcomes in the sense that they do not need to consider all possible \(F\)-feasible \((S, G)\). Rather, as in the complete-information correlated equilibrium (Aumann (2013)), the Bayes correlated equilibria are simply character-
ized by a number of inequalities that correspond to the obedience conditions for the agents to follow the “mediator’s recommendation” (in this sense, their approach treats additional information in an “implicit” manner). In mechanism design, there is a difficulty in treating correlated equilibrium as a solution concept: as far as I know, we cannot apply the revelation principle (at least in a straightforward manner) to focus on direct mechanisms in seeking optimal mechanisms. Therefore, despite its usefulness of their concept in prediction, we take another, more “explicit” way to predict the agents’ behavior given each specific \((S, G)\), and to identify \((S, G)\) that corresponds to the worst-case scenario for the designer. This explicit approach allows us to apply the standard revelation principle for Bayesian mechanisms.\(^\text{12}\)

### 3.1 Revenue guarantee by efficient mechanism

First, we study the maximum expected revenue that can be guaranteed among all efficient auction mechanisms. Such a question may be relevant when the designer is a public entity selling its asset: its primary concern may be to allocate the asset in the most efficient way, but if there are multiple ways to sell the asset efficiently, the entity may desire to achieve higher revenue.\(^\text{13}\) If \(F\) satisfies independence, then it is the expected value of the second highest value, which is, for example, achieved by a second price auction. If \(F\) has certain correlation, then as in Crémer and McLean (1985, 1988), full surplus extraction is possible (and hence the expected revenue is the expected value of the highest value) if there is no additional information. However, if additional information is available to the bidders, full surplus extraction is no longer possible. Furthermore, we show that, if any

\(^{12}\)Of course, there may be an appropriate version of revelation principle for correlated equilibrium, which allows us to directly apply the Bayes correlated equilibria of Bergemann and Morris (2013). Whether such a version of revelation principle exists is an open question.

\(^{13}\)Note that, with a correlated prior and arbitrary additional information, the standard revenue equivalence result does not generally hold, and hence, there may be multiple efficient mechanisms with different revenue levels.
additional information is allowed, then given any \( F \) (independent or correlated), the highest expected revenue we can guarantee is the expected value of the second highest value. For example, such a level of expected revenue is guaranteed by a second price auction.

**Definition 4.** \( \Gamma = \langle M, q, p \rangle \) is efficient given \( F \) if, for any \((S, G)\) that is \( F \)-feasible, there is a Bayesian equilibrium \( \sigma^* \) such that, for each \( v, s \),

\[
\sum_{i|v_i = v^{(1)}} q_i(\sigma^*(v, s)) = 1.
\]

Given any \( F \), the set of efficient mechanisms is nonempty, because a second price auction is efficient. Moreover, a second price auction guarantees expected revenue \( E(v^{(2)}) \equiv \int v^{(2)}dF \), because given whatever \((S, G)\) that is \( F \)-feasible, truth-telling is each bidder’s weakly dominant action, and so it is always a Bayesian equilibrium that everyone reports his value truthfully.

We show that \( E(v^{(2)}) \) is indeed the highest revenue we can guarantee among all efficient mechanisms.

**Theorem 1.** \( E(v^{(2)}) \) is the highest expected revenue we can guarantee among all efficient mechanisms.

*Proof.** We consider the following information structure. Given each \( v \), distributed according to \( F \), if there is \( i \) such that \( v_i > v^{(1)}_{-i} (= \max_{j \neq i} v_j) \), then each agent observes \((i, v_{-i})\) as a public (among the agents) information.

More precisely, we let \( S_j = \{1, \ldots, N\} \times [0,1]^{N-1} (\ni (i, v_{-i})) \) for each \( j \), and \( G \) be the following. Let \( \delta(s) \in \Delta(S) \) be a Dirac measure for \( s \in S \) (i.e., for any measurable \( A \subseteq S \), \( \delta(A) = 1 \) if \( s \in A \), and \( \delta(A) = 0 \) if \( s \notin A \)). Then, for each measurable \( B \subseteq V \times S \),

\[
G(B) = \sum_{i=1}^{N} \int_{(v,s)\in B} 1\{v_i > v^{(1)}_{-i}\}d\delta((i, v_{-i}), \ldots, (i, v_{-i}))dF.
\]

In this information structure, \((i, v_{-i}) \in S_j\) indicates \( i \) is the highest-value bidder, \( v_{-i} \) are all the losers’ values, and this signal is common knowledge.

\[\text{14We ignore ties without loss of generality.}\]
among the bidders. Thus, the only asymmetric information among the bidders is the highest bidder $i$’s value, $v_i$, which is known solely to $i$.

To characterize the maximum revenue among all efficient mechanisms, we first consider the following “relaxed” problem. Imagine that the designer can also observe $(i, v_{-i})$, but not $v_i$ (except that $v_i > v_{-i}^{(1)}$). Then, in an efficient mechanism, $i$ must always win conditional on $s$, which implies that the maximum payment the mechanism can charge is $v_{-i}^{(1)} (= v^{(2)})$. The maximum expected revenue in this relaxed problem is then

$$\int_{v} v^{(2)} f(v) dv = E(v^{(2)}).$$

In the original problem, because the designer does not observe $(i, v_{-i})$, such a payment level provides an upper bound for the achievable revenue. However, this upper-bound level of expected revenue is guaranteed by a second price auction, regardless of any $(S, G)$ that is $F$-feasible. □

### 3.2 Approximate revenue guarantee

Next, we study the maximum expected revenue that can be guaranteed among all (not necessarily efficient) mechanisms. As opposed to the first result, our result does not characterize the maximum expected revenue that can be guaranteed, but we obtain an upper and a lower bound of it, where, under certain conditions on $F$, the difference of these bounds is exponentially small with respect to the number of bidders. A lower bound is again given by a second price auction (without a reserve). Thus, the result implies that a second price auction is “close to” the optimum, with a fast speed of convergence.

To facilitate understanding of the result of this section, first, suppose that $v$ is independently and identically distributed. Suppose further that the virtual value for each $i$, $v_i - \frac{1 - H(v_i)}{h(v_i)}$, is strictly increasing, where $H, h$ are the cdf and pdf for $v_i$. Then, as in Myerson (1981), the revenue-maximizing mechanism is a second price auction with a reserve price. Notice, however,
that the difference in expected revenues between second-price auctions with
and without a reserve is exponentially small with respect to $N$. This is
because the only event where some revenue difference arises is when every
bidder except for one has a value below the reserve, say $r \in (0, 1)$ (and in
that case, a second price auction with a reserve yields the revenue equal to
$r$, while a second price auction without a reserve yields the revenue equal to
$v^{(2)}(< r)$), while such event occurs with probability $N \Pr(v_i < r)^N \Pr(v_i >
F_{r})$. Therefore, for the expected revenue, there exist $\alpha, \beta > 0$ such that
$E(v^{(2)}) \geq E(R) - \alpha e^{-\beta N}$, i.e., the difference $E(R) - E(v^{(2)})$ is vanishing at
an exponential rate.$^{15}$

The goal of this section is to show that, even if $v$ is correlated according
to $F$, we obtain a similar result under certain conditions. We now introduce
the key assumption of this section.

**Assumption 1.** There exists a profile of (i) a measurable set $\Theta$, (ii) $\mu \in
\Delta(\Theta)$, and (iii) $H_\theta \in \Delta([0, 1])$ for each $\theta \in \Theta$, that satisfies the following.

(I) For any measurable $A_i \subseteq [0, 1]$ for $i$,

$$F(\prod_{i=1}^{N} A_i) = \int_{\Theta} \prod_{i=1}^{N} H_\theta(A_i) \, d\mu.$$ 

(II) For each $\theta \in \Theta$, $H_\theta \in \Delta([0, 1])$ has a density $h_\theta$, and that each $i$’s
virtual valuation $v_i - H_\theta(v_i)$ is strictly increasing in $v_i$.

The assumption says that, even though we do not $\textit{a priori}$ assume an
independent distribution, $F$ can be written as a mixture of conditionally
independent and identical distributions that satisfy Myerson’s “regularity
condition”. This is admittedly a demanding assumption. For example, this
requires symmetry of the distribution of $F$, and excludes negative correlation
in \( F \). Nevertheless, as de Finetti theorem shows, whenever \( F \) is exchangeable (which is a common assumption in applied probability theory with large samples), it can always be written as a mixture of conditionally independent and identical distributions. Moreover, under a mild condition on the underlying distribution, Myerson’s regularity condition for \( H_\theta \) is guaranteed. In this sense, we believe that our assumption is reasonable. See the appendix for a more formal relationship between our Assumption 1 and exchangeability.

**Theorem 2.** The maximum expected revenue that is guaranteed among all mechanisms is no greater than \( E(v^{(2)}) + \int_\Theta \alpha(\theta)e^{-\beta(\theta)N}d\mu. \)

*Proof.* We consider the information structure such that each bidder can “observe” \( \theta \in \Theta \) as a public (among them) information, while each \( v_i \) is \( i \)'s private information.

More precisely, we let \( S_i = \Theta \) for each \( i \), and \( G \) be the following. Let \( \delta(s) \in \Delta(S) \) be a Dirac measure for \( s \in S \). Then, for each measurable \( A \subseteq V \times S \),

\[
G(A) = \int_{(v,s)} dH_\theta \delta(\theta, \ldots, \theta) d\mu.
\]

In this information structure, \( \theta \) indicates the “realized” value of the fundamental, which is common knowledge among the bidders. The values are then conditionally independently and identically distributed according to \( H_\theta \).

To identify an upper-bound revenue among all mechanisms, we first consider the following relaxed problem. Imagine that the designer can also observe \( \theta \), but not each \( v_i \). Given each \( \theta \), the problem is then a standard optimal auction problem with independently distributed values. The maximum expected revenue given \( \theta \) is achieved by a second price auction with the optimally set reserve price, say \( r_\theta \).

In the original problem, because the designer does not know \( s \), the maximum expected revenue we can guarantee is no greater than \( E(v^{(2)}) + \int_\Theta \alpha(\theta)e^{-\beta(\theta)N}d\mu. \)

\( \square \)
3.3 Revenue guarantee in sequential sales

The third application considers a sequential-sales setting. Consider a discrete-time \((t = 1, 2, \ldots)\) dynamic environment where each agent \(i\) is available only at time \(t = i\). The designer (a seller) has an indivisible good, and attempts to find the timing of selling it in order to maximize expected revenue. Although our model introduced in Section 2 is static, we treat this dynamic environment as an application of our model by requiring that a feasible mechanism should be such that the allocation for agent \(i\) given a message profile \(m = (m_1, \ldots, m_N), (q_i(m), p_i(m))\), can only be a function of \(m^i \equiv (m_1, \ldots, m_i)\) but not of \(m_{i+1}, \ldots, m_N\).

Because the allocation for agent 1 can only be a function of his own message, it is in general impossible to extract the “full” surplus. However, if \(F\) exhibits certain correlation as in Crémer and McLean (1985), Crémer and McLean (1988), or McAfee and Reny (1992), and if we do not allow for any additional information for the agents, then no information rent would be left for all of the other agents \(i \neq 1\). In this sense, much of the surplus could still be extracted by a similar side-bets mechanism as in those papers.

If the agents may have additional information, however, then this surplus extraction result does no longer hold. Furthermore, we show that the highest revenue guarantee is attained by a dominant-strategy mechanism. More specifically, it is a sequential posted-price mechanism where the price for each agent \(i\) is adjusted by the previous agents’ reports, \(v_1, \ldots, v_{i-1}\). Therefore, this is an instance where the revenue-maximizing seller finds it optimal (exactly rather than approximately) to use a dominant-strategy mechanism.

First, we characterize the optimal dominant-strategy mechanism, which is obtained by a backward-induction argument. Let \(v^i = (v_1, \ldots, v_i)\). First, consider the last buyer, \(i = N\). If each agent \(j < N\) has been assigned the good with probability \(q_j\), then the optimal dominant-strategy mechanism
solves

\[ R_N(q_1, \ldots, q_{N-1}) = \max_{q_N(v^N), p_N(v^N)} E(q_N(v^N)p_N(v^N)) \]

\text{sub. to} \quad q_N(v^N)(v_N - p_N(v^N)) \geq q_N(v^{N-1}, v'_N)(v_N - p_N(v^{N-1}, v'_N)),

q_N(v^N)(v_N - p_N(v^N)) \geq 0,

\sum_{j=1}^{N-1} q_j + q_N(v^N) \leq 1.

By an induction argument, let \( R_{i+1}(q_1, \ldots, q_i) \) be the expected revenue from the agents \( k = i+1, \ldots, N \) in the optimal dominant-strategy mechanism given that each agent \( j \leq i \) has been assigned the good with probability \( q_j \). Now, regarding the allocation for agent \( i \), the optimal dominant-strategy mechanism solves

\[ R_i(q_1, \ldots, q_{i-1}) = \max_{q_i(v^i), p_i(v^i)} E(q_i(v^i)p_i(v^i) + R_{i+1}(q_1, \ldots, q_i(v^i))) \]

\text{sub. to} \quad q_i(v^i)(v_i - p_i(v^i)) \geq q_i(v^{i-1}, v'_i)(v_i - p_i(v^{i-1}, v'_i)),

q_i(v^i)(v_i - p_i(v^i)) \geq 0,

\sum_{j=1}^{i-1} q_j + q_i(v^i) \leq 1.

Therefore, we obtain the following.

\textbf{Lemma 2.} The expected revenue of the optimal dominant-strategy mechanism is given by \( R_1 \).

We now show that \( R_1 \) is indeed the highest revenue guarantee.

\textbf{Theorem 3.} The highest revenue guarantee among all feasible mechanisms in the sequential-sales environment is \( R_1 \).

\textit{Proof.} We consider the information structure such that bidder \( i \) observes \( v^i = (v_1, \ldots, v_i) \). More precisely, we let \( S_i = V^i \equiv V_1 \times \cdots \times V_i \) for each \( i \),
and $G$ be the following. Let $\delta_j(s_j) \in \Delta(S_j)$ be a Dirac measure for $s_j \in S_j$. Then, for each measurable $A \subseteq V \times S$,

$$G(A) = \int_{(v,s) \in A} \prod_{i=1}^{N} d\delta_i(v^i) dF.$$ 

Suppose that there exists a feasible mechanism $\Gamma = (M, \tilde{q}, \tilde{p})$ and its Bayesian equilibrium $\sigma$ given this information structure that yields a strictly higher expected revenue than $R_1$.

In the following, we only consider the case where $\sigma$ is a pure-strategy equilibrium, but a similar logic applies to mixed-strategy equilibria as well. For each $i$ and $v$, let $q_i(v^i) = \tilde{q}_i(\sigma^i(v^i))$ and $p_i(v^i) = \tilde{p}_i(\sigma^i(v^i))$.

First, consider the last buyer, $i = N$. Because this buyer knows the realized $v^N$, the Bayesian equilibrium condition is that, for each $m_N \in M_N$,

$$\tilde{q}_N(\sigma_N(v^N), \sigma^{N-1}(v^{N-1}))(v_N - \tilde{p}_N(\sigma_N(v^N), \sigma^{N-1}(v^{N-1}))) \geq \tilde{q}_N(m_N, \sigma^{N-1}(v^{N-1}))(v_N - \tilde{p}_N(m_N, \sigma^{N-1}(v^{N-1}))),$$

which implies that, for each $v'_N \in V$,

$$q_N(v^N)(v_N - p_N(v^N)) \geq q_N(v^{N-1}, v'_N)(v_N - p_N(v^{N-1}, v'_N)).$$

Therefore, the expected revenue from this buyer is at most

$$\max_{q_N(v^N), p_N(v^N)} E(q_N(v^N)p_N(v^N))$$

subject to

$$q_N(v^N)(v_N - p_N(v^N)) \geq q_N(v^{N-1}, v'_N)(v_N - p_N(v^{N-1}, v'_N)),$$

$$q_N(v^N)(v_N - p_N(v^N)) \geq 0,$$

$$\sum_{j=1}^{N-1} q_j(v^j) + q_N(v^N) \leq 1,$$

which is $R_N(q_1(v^1), \ldots, q_{N-1}(v^{N-1}))$.

By an induction argument, suppose that $R_{i+1}(q_1(v^1), \ldots, q_i(v^i))$ is an upper-bound expected revenue from the agents $k = i + 1, \ldots, N$. Now,
regarding the allocation for agent $i$, because this agent knows the realized $v^i$, the Bayesian equilibrium condition is that, for each $m_i \in M_i$,

$$
\tilde{q}_i(\sigma_i(v^i), \sigma^{i-1}(v^{i-1}))(v_i - \tilde{p}_i(\sigma_i(v^i), \sigma^{i-1}(v^{i-1}))) \\
\geq \tilde{q}_i(m_i, \sigma^{i-1}(v^{i-1}))(v_i - \tilde{p}_i(m_i, \sigma^{i-1}(v^{i-1}))),
$$

which implies that, for each $v'_i \in V$,

$$
q_i(v^i)(v_i - p_i(v^i)) \geq q_i(v^{i-1}, v'_i)(v_i - p_i(v^{i-1}, v'_i)).
$$

Thus, the expected revenue is at most

$$
\max_{q_i(v^i), p_i(v^i)} \quad E(q_i(v^i)p_i(v^i) + R_{i+1}(q_1, \ldots, q_i(v^i))) \\
\text{sub. to} \quad q_i(v^i)(v_i - p_i(v^i)) \geq q_i(v^{i-1}, v'_i)(v_i - p_i(v^{i-1}, v'_i)), \\
q_i(v^i)(v_i - p_i(v^i)) \geq 0, \\
\sum_{j=1}^{i-1} q_j + q_i(v^i) \leq 1,
$$

which is $R_i(q_i(v^1), \ldots, q_{i-1}(v^{i-1}))$.

Therefore, the expected revenue is at most $R_1$, which contradicts our initial supposition. \qed

4 Conclusion

In this paper, we consider auction environments with a (possibly correlated) common prior over bidders’ values, where each bidder may have additional information (e.g., through information acquisition). We characterize the optimal mechanisms in terms of the expected revenue that is guaranteed given whatever additional information is available to the bidders. Specifically, we show that (i) a second-price auction is optimal among all the efficient mechanisms, (ii) it is “approximately” optimal among all the mechanisms, and (iii) a sequential posted-price mechanism is optimal in certain dynamic environments.
Although the paper focuses on simple auction environments, I believe that the idea of decomposing a (possibly correlated) distribution into multiple independent distributions (with the interpretation of the agents’ additional information) is useful in other mechanism design contexts.

A Assumption 1 and Exchangeability

In Section 3.2, we show the approximate optimality of a second-price auction under Assumption 1, a possibility of representing $F$ as a mixture of conditionally independent and identical distributions. As we discuss there, Assumption 1 is related to exchangeability, a common assumption in applied probability theory with large samples. Here, we provide a more formal relationship between Assumption 1 and exchangeability.

Recall that exchangeability is about the distribution of a countably infinite sequence of random variables. Let $F_N \in \Delta([0,1]^N)$ denote a probability distribution over a countably infinite sequence of random variables, each taking a value in $[0, 1]$. A possible (though not necessary) interpretation is that there are potentially infinitely many bidders $i = 1, 2, \ldots$, and $F_N$ provides a (possibly correlated) joint distribution over the values of all those bidders. Consider any finite permutation of identities of the bidders. Let $\tilde{F}_N$ denote the joint distribution over the values of all the bidders, but with the permuted identities. We say that $F_N$ is exchangeable if $F_N = \tilde{F}_N$ for every finite permutation. With abuse of terminology, we say that $F$ (a distribution over $N$ bidders’ values) is exchangeable if it can be expressed as a marginal of an exchangeable $F_N$ on $N$ variables (note that, by exchangeability, “which” $N$ variables does not make any difference).

The following theorem, called de Finetti theorem, says that exchangeable $F_N$ can be represented as a mixture of conditionally independent and identical distributions.

**Lemma 3.** (de Finetti theorem) If $F_N$ is exchangeable, then there exists a profile of (i) a measurable set $\Theta$, (ii) $\mu \in \Delta(\Theta)$, and (iii) $H_\theta \in \Delta([0,1])$ for
each \( \theta \in \Theta \), such that, for any measurable \( A_i \subseteq [0,1] \) for \( i \in \mathbb{N} \), we have

\[
F_N(\prod_{i=1}^{\infty} A_i) = \int_{\Theta} \prod_{i=1}^{\infty} H_\theta(A_i) \, d\mu.
\]

Therefore, Property (I) in Assumption 1 is obtained by assuming exchangeability of \( F \).

Next, for each \( m \geq 2 \) and \( v_1, \ldots, v_{m-1} \in [0,1] \), let \( P(\{v_m \in \cdot\}|v_1, \ldots, v_{m-1}) \) denote the conditional probability measure (induced by \( F_N \)) for \( v_m \) given realization of \( v_1, \ldots, v_{m-1} \). This \( P(\{v_m \in \cdot\}|v_1, \ldots, v_{m-1}) \) is called a *predictive measure*. It is known that \((\mu-)\text{almost every} H_\theta \) can be obtained as the limiting distribution of some sequence of predictive measures as \( m \to \infty \) (and vice versa).\(^{16}\) Therefore, Property (II) in Assumption 1 (or more precisely, its \("(\mu-)\text{almost every}\" version) is implied by imposing the corresponding condition on \((F_N-)\text{almost every} \) predictive measure.

The discussion in this section is summarized in the following proposition.

**Proposition 1.** Let \( F_N \in \Delta([0,1]^N) \) be exchangeable, and for every \( m \geq 2 \) and \((F_N-)\text{almost every} v_1, \ldots, v_{m-1} \), predictive measure \( P(\{v_m \in \cdot\}|v_1, \ldots, v_{m-1}) \) satisfies Myerson’s regularity condition, i.e., it admits a density with which the virtual value is strictly increasing. If \( F \in \Delta([0,1]^N) \) is induced by exchangeable \( F_N \), then there exists a profile of (i) a measurable set \( \Theta \), (ii) \( \mu \in \Delta(\Theta) \), and (iii) \( H_\theta \in \Delta([0,1]) \) for each \( \theta \in \Theta \), that satisfies the following.

(I) For any measurable \( A_i \subseteq [0,1] \) for \( i \),

\[
F_N(\prod_{i=1}^{N} A_i) = \int_{\Theta} \prod_{i=1}^{N} H_\theta(A_i) \, d\mu.
\]

(II) For \((\mu-)\text{almost every} \theta \in \Theta \), \( H_\theta \in \Delta([0,1]) \) has a density \( h_\theta \), and that each \( i \)'s virtual valuation \( v_i - \frac{1-H_\theta(v_i)}{h_\theta(v_i)} \) is strictly increasing in \( v_i \).

\(^{16}\)For example, see Berti, Mattei, and Rigo (2002).
Hence, we obtain the conditions in Assumption 1. Though redundant, we note that the same result is obtained under those alternative assumptions.

**Theorem.** The maximum expected revenue that is guaranteed among all mechanisms is no greater than $E(v^{(2)}) + \int_{\Theta} \alpha(\theta)e^{-\beta(\theta)N}d\mu$.

**References**


