The Inevitability and Ubiquity of Cycling in All Feasible Legal Regimes: A Formal Proof

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Abstract

Intransitive choices, or cycling, are generally held to be the mark of irrationality. When a set of rules engenders such choices, it is usually held to be irrational and in need of reform. In this article, we prove a series of theorems, demonstrating that all feasible legal regimes are going to be rife with cycling. Our first result, the Legal Cycling Theorem, shows that unless a legal system meets some extremely restrictive conditions, it will lead to cycling. The discussion that follows, along with our second result, the Combination Theorem, shows exactly why these conditions are almost impossible to meet. All of this has numerous implications, which we can here only allude to. For one, it suggests why law is as susceptible to manipulation and loophole exploitation as it has proved to be. It also casts doubt on many standard modes of argument within and about law, that depend on appeals to the need for transitivity, Kaplow and Shavell’s critique of fairness-based justifications of legal doctrines being a prominent recent example.

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1 Introduction

Cycling, or intransitivity, is generally viewed as the hallmark of irrationality. To make cyclical choices, the argument goes, is to be incoherent. In fact, many an argument in law is built on demonstrating that adopting certain rules leads to cycling, often through the invocation of an elaborate, clever, albeit unrealistic hypothetical. Perhaps the most fundamental recent argument in this literature was offered by Kaplow and Shavell as a challenge to fairness-based legal doctrines. They demonstrated that the Pareto principle is incompatible with fairness-based doctrines—under the assumption of no cycling. They concluded that we need to get rid of such fairness-based legal doctrines, unless, that is, we are willing to give up on the Pareto principle. Living with cycles they regarded as absurd.

What we seek to demonstrate, with the help of two formal theorems, and a series of illustrations involving familiar legal systems, is that all remotely feasible legal systems, and certainly all that are known to have existed, are riddled with cycles. Nor are they mere occasional pathologies; they are rampant. Our first theorem, the Legal Cycling Theorem, shows that all legal systems that are not what we call “lex-utilitarian,” are going to exhibit cycles. We then present a number of examples and considerations to substantiate the claim that we would not want a legal system that is lex-utilitarian. Our second theorem, the Combination Theorem, shows why a lex-utilitarian legal system is almost impossible to construct, even if one wanted to go to the trouble of creating one.

This paper is organized as follows: The next section introduces the formal notation and shows a full characterization of the legal systems that induce cycles (the Legal Cyclic Theorem.) The subsequent section proceeds to show that actual legal systems induce cycles. Section 4 shows that attempts to change laws and eliminate cycles either fail or have implications that are so perverse as to be unacceptable. Section 5 lays bare an alternative source of cycling in legal regimes, arising from the fact that combining legal doctrines in any plausible fashion often leads to cycling (the Combination Theorem.) Section 6 revisits some classic results, such as Sen’s Liberal Paradox and Kaplow and Shavell’s “Anti-Fairness” Theorem that take on a different significance in light of our results. Section 7 argues that, in addition to being ubiquitous and ineradicable, cycles are also problematic, among other things, because they create widespread opportunities for manipulation. However, the full development of this point is beyond the scope of this paper and left for future work. Section 8 discusses related literature on intransitivity in law. Section 9 concludes. Proofs are in the appendix.
2 The Legal Cycling Theorem

We begin by defining a very special kind of legal regime, which we call, for reasons that will soon become apparent, lex-utilitarian. We then show that a lex-utilitarian system is the only one that is guaranteed to avoid cycles. By contrast, a system that is not lex-utilitarian is guaranteed to exhibit cycling, and, conversely, a system that exhibits cycling is guaranteed not to be lex-utilitarian. In other words, a system is free of cycles if and only if it is lex-utilitarian. After that, we show why all feasible legal systems are bound not to be lex-utilitarian—and therefore bound to exhibit cycling.

2.1 Lex-Utilitarian Legal Systems

Let $A$ be a finite set of alternatives. An issue $B$ is a subset of $A$ with two distinct elements. So, if $x \in A$ and $y \in A$, $x \neq y$, then the subset $B = \{x, y\}$ of $A$ is an issue. Let $\mathcal{B}$ be the set of issues. A legal system is a mapping $\mathcal{L}$ that takes an issue $B$, as input, and returns, as output, a non-empty subset, $\mathcal{L}(B)$, of $B$. So, a legal system is a mapping $\mathcal{L} : \mathcal{B} \rightarrow \mathcal{B} \cup A$ such that $\emptyset \neq \mathcal{L}(B) \subseteq B$. If there are two available options $x$ and $y$ then both of them may be legal or only one of them may be legal. The legal system $\mathcal{L}$ determines which options are legal: $\mathcal{L}(\{x, y\})$ are the legal alternatives when the available options are $x$ and $y$. If $\mathcal{L}(\{x, y\}) = \{x, y\}$ then both $x$ and $y$ are legal. If $\mathcal{L}(\{x, y\}) = \{x\}$ then only $x$ is legal. In section 9, these definitions, and corresponding results, are extended to choices with more than two options. Let $\mathcal{R}$ be the set of real numbers.

Definition 1 A legal system $\mathcal{L}$ is lex-utilitarian if there is an utility function $u : A \rightarrow \mathcal{R}$ such that

- $\mathcal{L}(\{x, y\}) = \{x, y\}$ if $u(x) = u(y)$;
- $\mathcal{L}(\{x, y\}) = \{x\}$ if $u(x) > u(y)$.

In other words, a legal system is lex-utilitarian if there is a function that ranks all theoretically possible alternatives from top to bottom, and if someone, choosing from a feasible subset of all these options, is obliged to choose the highest-ranking one. An option outranked by another is illegal. The highest-ranking choices are not necessarily unique. If two alternatives have the same (top-)rank, then they are both legal.

2.2 The Law-Abiding Citizen

A law abiding citizen is a rational agent who is constrained by the law. Let $P$ be the preference order (i.e., a complete and transitive binary relation over all alternatives) of a law
abiding citizen. It ranks all feasible alternatives from top to bottom and $x P y$ denotes a preference for $x$ over $y$. We also assume that $P$ is asymmetric. This rules out indifference between alternatives and, so, rules out “spurious cycles” where, for example, the decision maker is indifferent between three legal alternatives and chooses them in a cycle. In Section 9 we show our results when indifference is allowed.

A choice function $C$ is a mapping that takes an issue $B$, as input, and returns, as output, an element $C(B) \in B$. So, a choice function is a mapping $C : B \to A$ such that $C(B) \in B$. The law abiding citizen chooses the best option among the legal ones. So, given the law abiding citizen asymmetric preference order $P$ and the legal system $\mathcal{L}$, the law abiding citizen choice function $C := C_{P,\mathcal{L}}$ is such that

$$C_{P,\mathcal{L}}(B) \in \mathcal{L}(B) \text{ and } C_{P,\mathcal{L}}(B) P y \text{ for every } y \in \mathcal{L}(B), \ y \neq C_{P,\mathcal{L}}(B).$$

Hence, $C_{P,\mathcal{L}}(B)$ optimizes $P$ on $\mathcal{L}(B)$. When it is clear that we refer to the choice function of the law abiding citizen (and not an arbitrary choice function), we may drop the subscript $P, \mathcal{L}$ to ease the notation. So, between $x$ and $y$, the choice of the law abiding citizen is $x$, (i.e., $C(\{x, y\}) = x$) if and only if either $x$ is the only legal alternative (i.e., $\mathcal{L}(\{x, y\}) = \{x\}$) or both alternatives are legal (i.e., $\mathcal{L}(\{x, y\}) = \{x, y\}$) and $x$ is preferred over $y$ (i.e., $x P y$).

The choices of law-abiding citizens are based on two principles. Law-abiding citizens are completely rational and order all options with strict preferences. This avoids spurious cycles arising from uninteresting reasons such as indifferences or cyclical preferences on the part of the citizen. Moreover, law-abiding citizens respect the law and do not choose illegal options. They pick their top-ranked option among those that are feasible and legal. We now turn to the question of whether the choices of a law-abiding citizen can be cyclic.

**Definition 2** A choice function $C$ is cyclic if there exist three distinct alternatives $x$, $y$ and $z$ such that

$$C(\{x, y\}) = x; \ C(\{y, z\}) = y; \ C(\{x, z\}) = z.$$

The cycle of length 3 is without loss of generality because if there are cycles of any length then there must also be one of length 3.

**Definition 3** A legal system $\mathcal{L}$ induces cycles if there exists a preference $P$ such that the resulting choice function $C_{P,\mathcal{L}}$ of a law abiding citizen is cyclic.

We speak of a legal system inducing cycles to make it clear that we are dealing with perfectly rational decision-makers. If there were no law, as in the special case of a legal system that makes all options legal, there would be no cycles.
2.3 Main Result

**Legal Cycling Theorem** Consider the case where there are at least three distinct alternatives. Then,

(a) No lex-utilitarian legal system induces cycles.

However,

(b) Any legal system that is not lex-utilitarian induces cycles.

**Proof:** See Appendix.

The Legal Cycling Theorem is a full characterization of the legal systems that induce cycles. The upshot of the theorem is this: A legal system needs to tell a citizen which among a set of options he faces is legal. One possible way of picturing his situation is the lex-utilitarian set-up. Namely, we assume that there is a ranking of all possible options citizens might face and we picture the legal system as requiring a citizen to choose the highest-ranked option among those available to him. If several are ranked equally highly, he gets to choose among them as he pleases, i.e., in accordance with his preferences. It is probably fairly intuitive that this type of system, which so closely resembles the usual choice situation in economics, will not lead to cycling. What seems less intuitive and more interesting is the second part of the Legal Cycling Theorem—that a legal regime that is not susceptible to cycling will necessarily be capable of being reduced to this picture, in other words, that unless it can be thought of in lex-utilitarian terms, it will necessarily exhibit cycles.

3 Cycling in Actual Legal Regimes

Let us now look at examples of cycles as they arise under the common law. We look at four legal doctrines and some cycles they each can generate: duress, self-defense, necessity and negligence. These doctrines are in no way peculiar to the common law. Every legal regime known to us, indeed every legal regime conceivable to us, has these doctrines. They seem to represent basic, culture-insensitive facets of human morality which legal regimes cannot but help reflecting. After presenting these doctrines and the cycles they give rise to, we explore two strategies that might suggest themselves for getting rid of the cycles. The first strategy, consistent with the Legal Cycling Theorem, fails and simply results in producing new and different cycles. The second strategy succeeds but, consistent with the theorem, renders the system lex-utilitarian. This second kind of failure is particularly important to understand because it reveals just why lex-utilitarian systems are not really feasible.
3.1 Cycling and Duress

The defense of duress is available to a defendant who was pressured into committing a crime with the threat of serious pain or injury. If, for instance, someone were threatened with being subjected to serious burns unless he helped out in a bank robbery, he would probably have the duress defense available to him. To be sure, the defense is not available unless the threat is sufficiently serious. Merely being threatened with something that one considers extremely disadvantageous is not enough. If someone is threatened with the destruction of a piece of property he greatly treasures, even a manuscript he has been working on for many years, that almost surely does not qualify.

The duress defense induces a cycle in the following way. Imagine that the defendant happens to value his manuscript so highly that if a fire were to break out that threatened to consume it, he would not hesitate to rush into the burning building, to salvage it, even at the cost of suffering serious burns. Now we get the following cycle: When choosing between letting the manuscript be destroyed or suffering burn wounds, the defendant will choose to suffer burn wounds. When choosing between suffering burn wounds or participating in the bank robbery, the defendant will choose to participate in the bank robbery, which is permitted by the duress defense. Alas, when choosing between participating in the bank robbery or seeing his treasured manuscript be destroyed by gangsters, he will choose to let his manuscript be destroyed—because that is what the law expects of him under the circumstances, there being no duress defense if he makes the contrary decision. Here then the legal system induces a cycle, that is, it produces intransitive choices in someone who makes rational decisions while subjecting himself to its rules.

The Legal Cycling Theorem implies that the root cause of this cycle is that the doctrine of duress does not render the legal system lex-utilitarian. The key property that any lex-utilitarian legal system must have, but actual legal systems do not have, is the following:

$$
L(\{x, y\}) = \{x, y\} \text{ and } L(\{y, z\}) = \{y, z\} \Rightarrow L(\{x, z\}) = \{x, z\}.
$$

Property (I) must be satisfied by any lex-utilitarian legal systems because for any function \(u\), \(u(x) = u(y)\) and \(u(y) = u(z)\) imply \(u(x) = u(z)\). However, in this example, option \(x\) is to participate in the bank robbery, option \(y\) is to suffer severe burns, and option \(z\) is to lose the manuscript. So, \(L(\{x, y\}) = \{x, y\}\) because under the doctrine of duress both the participation in the bank robbery and the enduring of severe burns are legal. In addition, \(L(\{y, z\}) = \{y, z\}\) because the choice between the burns or the manuscript concerns only the decision maker and both options are legal. Finally, \(L(\{x, z\}) = \{z\}\) because the duress defense does not apply to participation in the bank robbery if the alternative is to lose a
The defense of necessity is similar in structure but different in content from the defense of duress. It is available to someone who has a difficult choice to make and chooses to break the law rather than suffer, or inflict, some serious harm that is more serious than the harm that the law he is breaking is seeking to prevent. In other words, if he can do substantially more good than harm by breaking the law, he is permitted to do so. (Note that this is different from the duress defense which applies even when one is doing more harm than one is threatened with.) If, for instance, someone is hiking in the mountains and can only avoid starvation by breaking into a mountain cabin to help himself to its supplies, the defense of necessity would exonerate him. Like the defense of duress, necessity is only available if the injury being prevented by committing the offense is sufficiently serious. Suppose for instance that the hikers can only climb the mountain if they are first willing to break into a mountain cabin. In that case, the harm they would be avoiding by breaking into the mountain cabin would simply be that they would be deprived of the ability to climb the mountain. That would not be a serious enough harm for the necessity defense to apply.

To see what cycle this can generate, consider a hiker who is willing to risk starvation to climb the mountain. In other words, between avoiding the risk of starvation and climbing the mountain, he chooses to climb the mountain and risk starvation. If on climbing the mountain, he finds himself in the position where he can only avoid the risk of starvation by breaking into a mountain cabin, he breaks into the mountain cabin (and gets to avail himself
of the necessity defense). On the other hand, if confronted with the choice between breaking into a mountain cabin or foregoing his mountain climb, he chooses the latter (because he is law-abiding and does not get the benefit of the necessity defense). His choice between the three alternatives under consideration — break-in ($x$), possible starvation ($y$) and foregoing the mountain climb ($z$) — is thus intransitive.

The logic underlying the cycle induced by the necessity doctrine is the same as in the cycle induced by the duress doctrine. In both cases, the resulting law is not lex-utilitarian because property (I) does not hold. Instead, (N) holds mainly because the doctrine of necessity is also context-dependent. Whether or not the defense of necessity applies depends on what is done, it’s consequences and also on the available alternatives. The defense of necessity for breaking into the cabin (option $x$) holds if the alternative is $y$ (to risk starvation), but not if it is $z$ (to give up the mountain climb).

### 3.3 Cycling and Self-defense

To avoid getting seriously injured from someone’s attack on him, the defendant is allowed to seriously injure him in turn. He is not in general allowed to defend an attack on his property—e.g. his manuscript—by the use of “deadly force,” which refers to force that might seriously injure the attacker (as opposed to actually kill him). Now suppose that he is willing to incur serious injury to protect his manuscript from great harm. Once again we get a cycle. We would observe the defendant when choosing between getting injured or suffering damage to his manuscript, choosing to get injured instead. When choosing between getting injured or injuring his attacker, we would observe the defendant choosing the latter—injuring his attacker (as he is permitted to do by the doctrine of self-defense). When choosing between injuring someone who is about to destroy his manuscript, or permitting him to destroy the manuscript, he would choose the latter, because that is what the law of self-defense requires of him. In short, the doctrine of self-defense induced a cycle.

In this example, the cycle induced by the doctrine of self-defense has the same logical structure as the cycles induced by duress and necessity. Self-defense is also context-dependent. To use deadly force on the attacker ($x$) is legal if the alternative is to incur a serious injury ($y$), but not if it is to have the manuscript damaged ($z$). Moreover, if the options are limited to $y$ and $z$ then they are both legal. Thus, (N) holds. It follows that the legal system is not lex-utilitarian and if a law abiding citizen ranks $x$ above $y$ above $z$ then the resulting choices produce the cycle in (C).
3.4 Cycling and Negligence

The doctrine of negligence imposes liability on those who harm others through negligent actions. Negligence is generally understood to be the unjustifiable imposition of risk. Criteria of justifiability vary. A commonly invoked one is the “Hand formula:” Does the benefit of taking a precaution exceed its cost? Suppose the defendant is the innocent victim of a traffic accident. To avoid suffering permanent injuries, he speeds to the nearest emergency room, in the process imposing a risk in the amount of $pL$ (probability of loss times amount of loss) on the surrounding world. But if he had not done so, he would have suffered a risk in the amount of $p'L'$, much higher, let us say, than the one he imposed on others by speeding to the emergency room. Finally, let us suppose that he is an avid mountain climber who is prepared to run risks well in excess of $p'L'$ to get to a mountain top. This yields the by now familiar type of cycle. Given the choice between avoiding a risk in the amount of $p'L'$ and getting to the mountain top, he chooses the latter. Given the choice between getting to the mountain top and avoiding imposing a risk of $pL$ on others, he does not climb the mountain and avoids imposing a risk of $pL$ on others. Given the choice between avoiding imposing $pL$ on the others, and experiencing a risk $p'L'$, he chooses the latter, which once again yields a cycle.

Formally, this cycle is as follows: if $x$ is to impose a risk $pL$ on others, $y$ is to take a risk greater than $p'L'$ and $z$ is to give up the mountain climb. Then, just as in the examples involving duress, necessity and self-defense, $(N)$ holds, the legal system is not lex-utilitarian and induces the cycle in $(C)$.

Although the examples are very particular, they are constructed from a very general recipe that can be widely applied, which means that there is nothing rare or unusual about these cycles. The recipe is the following: There are a series of options which a decision-maker cares about to varying degrees—e.g., his manuscript, his physical safety, and not getting involved in a bank robbery. Each of these options has, in common moral and legal parlance, an interest associated with it, that is, in describing the situation we are led to refer to the decision-maker’s interest in his manuscript, his interest in his physical safety and the bank’s interest in not getting robbed. The relevant legal rules provide a ranking of these interests. They would generally put the bank’s interest in not being robbed ahead of the defendant’s interest in not having his manuscript destroyed; they would put the defendant’s interest in his physical safety ahead of his interest in his manuscript, and they would put the defendant’s interest in his physical safety ahead of the bank’s interest in not getting robbed. That’s of course a perfectly transitive ranking. What induces the cycle is that in choosing between the manuscript and his body, the defendant is allowed to choose what he prefers most, rather than what he has the greater legal interest in. Anytime we inject the possibility
of someone’s choosing what he has a lesser interest in, but greater desire for, over what he
has a greater interest in, but lesser desire for, a cycle like the above may result.

Using the conceptual framework of the proof of the legal cycle theorem we can appreciate
more clearly what gives rise to cycles. What we call interests correspond to a function that
ranks all options. So, if the options are: (a) his manuscript, (b) his physical safety, and (c)
not getting involved in a bank robbery, his interests rank b over c over a. However, the law
does not require him to always take the highest interest option. In this example, this is only
so in the case of the choice between a and c (where he is required to choose c). The other
choices are left to the decision maker. Hence, even if interests are perfectly ranked, legality
is not, at least on occasion, determined by the ranking of interests. Sometimes the decision
maker is allowed to choose an option with lower interest (e.g., b has higher interest than
a, but our law-abiding citizen chooses a over b). Thus, the law is not lex-utilitarian and,
thereby induces cycles.

4 Why Lex-Utilitarian Legal Systems are Unacceptable

It will prove illuminating to consider some of the strategies people might follow to try to
eliminate cycles. One strategy that probably suggests itself arises from a powerful, but as it
turns out, false intuition regarding the root cause of these cycles. It might seem for instance
that what generates the duress cycle is the law’s rigid assumption that physical safety is
always more precious than property. It might seem as though the cycle could be made to
disappear by simply making the law less rigid, or coarse-grained, by making the availability
of the duress defense depend not on the specific injury being threatened but on the amount
of disutility associated with the injury. Thus, one might say that because the loss of the
treasured manuscript is as serious to this particular defendant as physical injury is to most
other people, he gets to invoke duress when it is being threatened. Correspondingly, one
might say that because his physical safety is less precious to him than it is to other people,
he does not get to invoke the duress defense when that is what is being threatened. So long
as the defendant chooses the manuscript over physical safety, it seems as though the cycle
has now been made to disappear.

Alas, a closely related cycle can still be constructed. Suppose the defendant has
the choice between doing something which puts his manuscript at risk or puts his body at
risk. Inasmuch as his manuscript is more precious to him than his body, we would expect
him to put his body at risk. However, that does not take into account the effect which the
legal rules have on his decision. Inasmuch as he is entitled to protect his manuscript much
more extensively than he is entitled to protect his body—that is, he is entitled to participate in a bank robbery to avoid its destruction—this might well lead him to choose to put his manuscript at risk rather than his body. The cycle has now been recreated. It should be apparent that an analogous argument can be made about each of other cycles laid out. If we tried to modify the doctrines of self-defense, necessity and negligence by reformulating the law in terms of disutility rather than specific objects (like the body or property), a similar reformulation of the cycle is possible. This is just a special case of the familiar phenomenon of someone making himself more vulnerable because that entitles him to certain special benefits.

Let us now see what happens if we try to eliminate cycles through a different approach. More concretely, let us try to turn our cycle-prone legal system into one that is lex-utilitarian. As shown by the legal cycling theorem, this is the only strategy that can effectively eliminate cycles. However, it has extremely unattractive implications. In this sense, this is the more important strategy to explore because it helps reveal why lex-utilitarian legal systems are not really acceptable.

To be lex-utilitarian, it has to be the case that whenever we allow a decision maker to choose between various alternatives, they have to be “on a par” as far as the lex-utilitarian function is concerned. The lex-utilitarian function ranks all alternatives, and requires the decision maker to choose among the highest ranked available options. He gets to choose only if there are several equally ranked options.

Now let us imagine the following. The decision maker faces certain alternatives \( x, y, \) and \( z \). Each of these alternatives carries certain costs and benefits with it—pros and cons, that is, but only for him. They have no effect on anyone else. Presumably we would want him to be able to choose between these. We would in general want a system to allow him to choose among alternatives that affect him only, at least in general, maybe not invariably. Paternalism and other considerations might impose some limitations, but we do not require that all choices that produce negligible effects on others to be legal. We only require that some of these choices be legal. Assuming that we want this to be the case, then, in order for the system to be lex-utilitarian (and cycle-free) these options have to be deemed to be “on a par” as far as the lex-utilitarian ranking function is concerned. So, consider a choice between \( x \) and \( y \) and assume that, if these are the available options, they are both legal. Then, if the law is lex-utilitarian they must be equally ranked—or the decision maker cannot freely choose among them.

Next let us picture a situation in which a further option \( w \) is injected. This option has significant consequences to others, or rather: choosing \( w \) means sparing that other person certain risks or costs. This is the typical kind of situation contemplated by the negligence doctrine: either the defendant does what generates certain benefits to him (option \( x \)) or he
does what avoids the risk to others, but deprives him of his benefits (option \( w \)). Unlike the choice between \( x \) and \( y \), the choice between \( x \) and \( w \) does have consequences to others. In the latter case, to choose \( x \) means to reject \( w \) and so to let another incur certain risks. Let us suppose that he would be permitted to choose \( x \) over \( w \). Presumably that would be based on some sort of comparison between the benefits to him of \( x \), and the risks for the other person if he rejects \( w \).

Next let us suppose he faces the choice between \( y \) and \( w \). The \( y \)-option is associated with a different package of costs and benefits for our defendant. Depending on exactly how those costs and benefits compare with those associated with the \( w \)-option (which affect the other person in this set-up), we might or might not want to let the defendant choose \( y \) over \( w \). And yet, if the system is lex-utilitarian, he must treat \( x \), \( y \) and \( z \) as equivalents. If we allow him to choose \( x \) over \( w \), we must also allow him to choose \( y \) over \( w \) and \( z \) over \( w \), or we give rise to cycles. Hence, if \( x \) is, say, worth a million dollars to the decision maker and \( y \) is, say, worth one dollar and \( z \), a negative amount then, when the alternative is to spare someone some risks, either he is allowed to do any one of these or none of them. In sum, the negligence doctrine would have to be insensitive to the degree of benefit an option has for the decision maker when determining whether he is allowed to choose it, which seems bizarre.

Nothing in this hinges on the particular doctrines being considered. The doctrines of necessity, self-defense and duress, if they were to be made lex-utilitarian, would have to become equally insensitive to crucial attributes of an option, when determining legality. Virtually all sensible legal doctrines one can think of involves comparing option \( w \) with option \( x \) (if those are the available ones) on some basis or other, to decide which the defendant is entitled to choose; they will involve comparing option \( w \) with option \( y \) (if those are the available ones) to decide which the defendant is entitled to choose among the two; and they will come to different conclusions if \( x \) and \( y \) are sufficiently different. This, however, is precluded if doctrines are to become lex-utilitarian.

5 Why Lex-Utilitarian Systems are Impossible, or Nearly So.

In this section, we show the difficulty of combining two or more doctrines to produce a lex-utilitarian system, even if neither doctrine by itself induces cycling. The only significant pre-condition of our result is one we call doctrinal unanimity. That is, when all doctrines agree on which options should be legal, the legal system must do what they all agree on, rather than the opposite. While not restricted to this case, the difficulty we lay bare is
a particularly interesting phenomenon when there is no direct inconsistency between the doctrines being combined. Let us suppose they concern themselves with different subjects, and are in full agreement to the extent that they overlap in what they cover. In other words, where one doctrine applies, the other doctrine either does not apply, or if it does, produces the same result. Nevertheless when they are combined, they induce cycles.

Let’s start with the observation that an individual doctrine may not be applicable on all issues. For example, doctrines regarding copyrights infringements may not be applicable to determine legality on issues regarding the use of deadly force. Henceforth, a doctrine \( D \) is a mapping \( D : B \rightarrow B \cup \{n/a\} \) such that, for every \( B \in B \), if \( D(B) \neq n/a \) then \( \emptyset \neq D(B) \subseteq B \); moreover, \( D(B) \neq n/a \) for some issue \( B \). The expression \( D(B) = n/a \) refers to the case where the doctrine \( D \) is non-applicable and so, silent over which options are legal on the issue \( B \). If \( D(B) \neq n/a \) then the doctrine is applicable and expresses a viewpoint on the legality of different options when \( B \) are the feasible choices. In this case, \( D(B) \) are the options that doctrine \( D \) deem legal.

**Definition 4** A doctrine \( D \) is **conditionally lex-utilitarian** if there exists an utility function \( u : A \rightarrow \mathbb{R} \) such that whenever \( D(B) \neq n/a \),

\[
D(\{x,y\}) = \{x,y\} \quad \text{if} \quad u(x) = u(y); \quad (E)
\]

\[
D(\{x,y\}) = \{x\} \quad \text{if} \quad u(x) > u(y). \quad (H)
\]

Like a lex-utilitarian legal system, a lex-utilitarian doctrine also ranks all possible alternatives and, whenever the doctrine is applicable, an option outranked by another feasible one is illegal. If two alternatives have the same (top-)rank, then they are both legal, provided that the doctrine is applicable. Legality here refers, naturally, to the viewpoint expressed by the doctrine and not by the final legal system.

We assume that doctrines are conditionally lex-utilitarian in this section. This assumption is not necessary for our main result which holds even if we make no assumptions on the doctrines. However, restricting the doctrines to be conditionally lex-utilitarian makes the results clearer for the following reason: Let’s say that a legal system \( L \) **adopts** a doctrine \( D \) if \( L(B) = D(B) \) when \( D(B) \neq n/a \). So, a legal system adopts a doctrine when the legal system agrees with the doctrine, when the doctrine is applicable. If a legal system \( L \) adopts a doctrine \( D \) that is not conditionally lex-utilitarian then the legal system \( L \) is not lex-utilitarian (and so, induces cycles). This follows because for any utility \( u \), \( (E) \) and \( (H) \) cannot hold in the entire domain of issues, if it does not hold in the sub-domain of issues where the doctrine is applicable. Conversely, if a doctrine \( D \) is conditionally lex-utilitarian then some lex-utilitarian legal systems can adopt it. The adopting legal system can be directly constructed with the utility function \( u \) (of the conditionally lex-utilitarian doctrine \( D \)) and the
rules \((E)\) and \((H)\). Thus, if doctrines are restricted to be conditionally lex-utilitarian then no single doctrine, by itself, makes the final legal system not lex-utilitarian. So, no conditionally lex-utilitarian doctrine, by itself, necessarily induces cycles. This restriction makes clear that the difficulty in combining doctrines to construct a lex-utilitarian legal system is above and beyond the difficulty in making doctrines conditionally lex-utilitarian in the first place.

Let \(\mathcal{D}\) be the set of all doctrines and \(\mathcal{L}\) be the set of all legal systems. An aggregator \(\alpha\) is a function

\[
\alpha : \mathcal{D}^n \rightarrow \mathcal{L}
\]

that maps a profile of doctrines \((\mathcal{D}_1, \ldots, \mathcal{D}_n)\) into a legal system \(\mathcal{L}\).

**Definition 5** An aggregator \(\alpha\) maps conditionally lex-utilitarian doctrines into lex-utilitarian legal systems if the legal system \(\mathcal{L} = \alpha(\mathcal{D}_1, \ldots, \mathcal{D}_n)\) is lex-utilitarian whenever the doctrines, \(\mathcal{D}_1, \ldots, \mathcal{D}_n\), are all conditionally lex-utilitarian.

The key condition on \(\alpha\) is that it produces lex-utilitarian legal systems. As mentioned, the proviso that this only needs to be so when the doctrines themselves are conditionally lex-utilitarian makes the results stronger and clearer.

**Definition 6** An aggregator \(\alpha\) satisfies doctrinal unanimity if for any options \(x\) and \(y\), \(\mathcal{L}(B) = \mathcal{D}_k(B)\) whenever these three conditions hold: (i) \(\mathcal{L} = \alpha(\mathcal{D}_1, \ldots, \mathcal{D}_n)\); (ii) \(\mathcal{D}_k(B) \neq n/a\) for some \(k = 1, \ldots, n\); and (iii) \(\mathcal{D}_i(B) = \mathcal{D}_j(B)\) for all \(i = 1, \ldots, n\) and \(j = 1, \ldots, n\) such that \(\mathcal{D}_i(B) \neq n/a\) and \(\mathcal{D}_j(B) \neq n/a\).

So, an aggregator satisfies doctrinal unanimity if whenever all applicable doctrines agree on what the law should be on an issue then this is the final law on this issue. It may seem natural to assume that \(\mathcal{L}(B) = B\) if \(\mathcal{D}_i(B) \neq n/a\) for all \(i = 1, \ldots, n\). That is, when no doctrine is applicable then all options are legal. However, we do not need this assumption and do not make it.

**Proposition** Assume that there are three or more options and \(n \geq 2\) (so at least two doctrines must be aggregated into a final legal system). Then, no aggregator satisfies doctrinal unanimity and maps conditionally lex-utilitarian doctrines into lex-utilitarian legal systems.

**Proof:** See Appendix.

Under doctrinal unanimity, it is impossible to aggregate more than one doctrine and assure that the final legal system is lex-utilitarian. Hence, the sense that lex-utilitarianism
is infeasible is not physical impossibility. Rather, it is that more than one doctrine can potentially be used in the construction of a legal system. As long as there are two or more doctrines, it is not possible to aggregate them and assure lex-utilitarianism. This follows as long as the aggregation process satisfies doctrinal unanimity. No other conditions are required. The above Proposition regarding aggregation and the Legal Cycling Theorem then yields the ineradicability of cycles, resulting in the **Combination Theorem**.

**Combination Theorem** Assume that there are at least three options. If multiple doctrines are aggregated under doctrinal unanimity then it is impossible to ensure that the final legal system will not induce cycles.

Let us now illustrate the Combination Theorem with some examples. We offer three examples to illustrate this result. The first example is very abstract and schematic. Indeed it can be thought of as a slightly simplified version of our proof. The second is more concrete, but sufficiently generic to indicate that the result should be expected to apply in a wide variety of contexts.

Let us consider three possible alternatives $x$, $y$, and $z$, as indicated by the vortices in our diagram.

\[
\begin{array}{c}
  x \\
  \downarrow \downarrow \\
  y \quad z
\end{array}
\]

There is a doctrine $D_1$ that ranks $z$ above $x$, in other words, states that given the choice between $x$ and $z$, only $z$ is legal. The line running from $x$ to $z$, with the arrow pointing towards $z$, is meant to indicate that. The doctrine ranks $x$ and $y$ equally, meaning that in a choice between $x$ and $y$, it declares both to be legal. The lines running from $x$ to $y$, with arrows pointing towards both $x$ and $y$, is meant to indicate that. There is no line connecting $y$ and $z$ because the doctrine does not apply to that choice. $D_1$ could be made into a lex-utilitarian regime if we simply made it complete and transitive by drawing such a line between $y$ and $z$, with the arrow pointing towards $z$. That possibility is what makes $D_1$ conditionally lex-utilitarian.

Next let us consider doctrine $D_2$.

\[
\begin{array}{c}
  x \\
  \downarrow \downarrow \\
  y \quad z
\end{array}
\]

The lines between $y$ and $z$, with arrows pointing towards each, indicates that $D_2$ ranks $y$ and $z$ equally. In other words, if those two alternatives were to present themselves, $D_2$ would deem both legal. The lines between $y$ and $x$ indicate that $D_2$ ranks them both equally.
legal. No lines run between $x$ and $z$ because the doctrine is inapplicable to that choice. This doctrine too could be rendered completely transitive by extending it, namely by saying that according to $D_2$, in a choice between $x$ and $z$, both are legal. In other words, $D_2$ is conditionally lex-utilitarian.

What about combining the two doctrines, consistent with the principle of doctrinal unanimity? That would mean both $x$ and $y$ are legal (if both come up together), because the two doctrines agree on that (that is, according to doctrinal unanimity). If $y$ and $z$ were to present themselves together, both would be legal, because according to the only applicable doctrine, $D_2$, that would be true. On the other hand if $x$ and $z$ presented themselves together, only $z$ would be legal, because according to the only applicable doctrine, $D_1$, that would be true. Combining $D_1$ and $D_2$ consistent with doctrinal unanimity yields,

\[
\begin{align*}
  x &\rightarrow z \\
  \downarrow &\uparrow \\
  y
\end{align*}
\]

That is, property (I) does not hold. Instead, (N) holds. But that means that the legal regime that emerges from combining $D_1$ and $D_2$ is no longer lex-utilitarian and therefore (because of the legal cycling theorem) vulnerable to cycling.

Now let us make up a more concrete example by the simple strategy of filling $D_1$ and $D_2$ with specific doctrinal content. Let us suppose that $x$, $y$, and $z$ are three patients, any two of which might conceivably present themselves simultaneously for treatment in an emergency room, requiring the doctor in attendance to make a triage judgment as to whom to treat first. We will assume that this decision about priority of treatment really matters to the outcome. Let us assume, moreover, that their injuries are of roughly equal severity. Let us also assume that two of them, $x$ and $z$, were both involved in a boating accident, $x$ being the officer on that boat, and $z$ a mere passenger. Finally, let us assume that $y$ is also a ship officer, though not on the boat involved in this accident.

We could imagine there to be two choice doctrines relevant to this situation.

$D$—Special Duty ($D$—sp for short) provides as follows: “As between patients, where one of them owes a special duty to the other (as captains do to passengers, and doctors to patients, and so forth), the one who is owed the duty generally gets priority. Where both belong to the same professional class (e.g. both are doctors, or both are ship officers), priority is to be given according to needs and likelihood of benefitting from treatment.”

$D$—triage ($D$—t for short) provides as follows: “As between patients between whom no special relationship exists, or patients belonging to the same professional class (e.g. both are doctors, or both are ship officers), priority is to be given according to needs and likelihood of benefitting from treatment.”
Note that $D - sp$ and $D - t$ overlap a bit, as legal doctrines often do, although the area of overlap does not seem problematic because they provide for the same thing with regard to the contingency where they overlap (the case where both patients belong to the same professional class). Now let’s consider each doctrine a bit more closely.

Let’s take a closer look at $D - sp$. $D - sp$ would require that, as between $x$ and $z$, $z$ be treated ahead of $x$, since $z$ is a passenger and $x$ is an officer on the ship on which $z$ was injured, and therefore owes him a special duty. $D - sp$ would require that, as between $x$ and $y$, either could be treated first, since they are both ship officers. As for the choice between $y$ and $z$, $D - sp$ does not apply because there is no special duty and they do not belong to the same professional class.

$D - sp$ is conditionally lex-utilitarian because we could make it lex-utilitarian simply by requiring that in a choice between $y$ and $z$, $z$ should be the only legal alternative.

Let’s now take a closer look at $D - t$. $D - t$ would find that as between $x$ and $y$, both being ship officers, needs and likelihood of benefit should decide and since these are equal, choosing either $x$ or $y$ would be legal. As between $y$ and $z$, there being no special relationship between them, and they not belonging to the same professional class, needs and likelihood of benefit will decide. Since those are equal, giving priority to either $y$ or $z$ would be legal. As between $x$ and $z$, the doctrine would simply not apply, since there is a special relationship between them.

$D - t$ is also conditionally lex-utilitarian, because we could make it lex-utilitarian by simply requiring that in a choice between $x$ and $z$, both should be legal.

What happens if we combine both doctrines. In the choice between $x$ and $y$, the two doctrines agree that both should be legal, and therefore they both would be. In the choice between $y$ and $z$, the only applicable doctrine, $D - t$, declares both options to be legal, and therefore they both would be. Alas, in the choice between $x$ and $z$, the only applicable doctrine, $D - sp$, declares only $z$ to be legal. This means the combination $D - sp / D - t$ legal regime is not lex-utilitarian and therefore, according to the legal cycling theorem, vulnerable to cycles.

Our third example is meant to illustrate that the impossibility of combining doctrines to form a lex-utilitarian legal system holds regardless of whether, in each choice only one doctrine is applicable or both doctrines are applicable and one doctrine overrules another. Consider another case of triage in an emergency room. Once again there are three injured parties, $x$, $y$, and $z$. Let’s say that the injuries are sufficiently similar so that if the doctrine of negligence governs who is to receive priority from the doctor on duty, then he would be free to choose either $x$, $y$, or $z$. All three options are legal. Now let’s say that $x$ and $z$ happen to be husband and wife. The husband ($x$) dotes on his wife ($z$) and wants her to be treated ahead of him. Let’s say that freedom of contract overrules negligence in the
choice between treating the husband or the wife first: the wife must be treated ahead of the husband. However, in the choices between $y$ and $z$, and between $x$ and $y$, there is no contract among the parties and therefore negligence doctrine prevails, which means that as between them, the doctor is free to choose to treat either. This means that once again $(N)$ holds and the legal system is not lex-utilitarian.

5.1 Cyclic Law

There are two ways in which the law can induce cycles. One of them is when $(N)$ holds. That is, both options are legal in the choice between $x$ and $y$ and in the choice between $y$ and $z$, but only one option is legal in the choice between $x$ and $z$. This is the type of law-induced cycle that we have focused on up to now. It follows from an interaction between the law and the preferences of the decision maker. The other type of law-induced cycle is more direct and occurs when the law itself is cyclic. That is, when

\[ \mathcal{L}({\{x, y\}}) = \{x\}, \mathcal{L}({\{y, z\}}) = \{y\} \text{ and } \mathcal{L}({\{x, z\}}) = \{z\}. \]  

(CL)

In terms of our diagram, this occurs when

\[ x \rightarrow z \]

\[ \text{\ulcorner} \quad \downarrow \]

\[ y \]

In this case, the choices of a law-abiding citizen are cyclic regardless of the decision maker’s preferences. A law-abiding citizen must follow the law. So, if the law is cyclic then the choices of a law-abiding citizen must also be cyclic. Consider our proposition regarding aggregation of conditionally lex-utilitarian doctrines. In the proof of this proposition (see appendix) we show that, under unanimity, combined doctrines fail to produce a lex-utilitarian legal system because both $(N)$ and $(CL)$ can occur. So, when doctrines are combined the law become cyclic in some cases and also induces cycles in conjunction with the decision maker’s preferences in other cases, as in $(N)$. We now illustrate the case where combined doctrines make the legal system itself cyclic, with suitable variations on the examples above.

Let’s return to the triage in the emergency room, but now let’s assume that $x$’s injuries are far more serious than $y$’s injuries which are, in turn, far more serious than $z$’s injuries. By the negligence doctrine, we have a perfectly lex-utilitarian system, devoid of cycling problems. Party $x$ must be treated ahead of $y$ which must be treated ahead of $z$. If, however, as in our original example, $x$ and $z$ are husband and wife who agree that $z$ must be treated ahead of $x$ then, by freedom of contract, $z$ must be treated ahead of $x$. Hence, if the law is to follow negligence in the choices involving person $y$ (where no contract exists), and the principle of
freedom of contract (where a legal contract does exist) then we get the cyclic law where \( x \) must be treated ahead of \( y \) who must be treated ahead of \( z \) who must be treated ahead of \( x \).

This example is actually far more general than it might appear at first. One way of appreciating its generality is to replace the freedom of contract doctrine with another doctrine that formally accomplishes the same thing. For instance, let us change the facts a little. Let us no longer assume that \( x \) and \( z \) are husband of wife. Instead, let us assume that \( x \) attacked \( z \) and the injuries each got are the result of that fight. We might now plausibly adopt an “equitable consideration” doctrine that applies comparatively between \( x \) and \( z \) and gives \( z \) priority over \( x \). If we assume, as we plausibly might, that the relative priority decreed by the negligence doctrine between \( x \) and \( y \) and between \( y \) and \( z \) remains untouched, we get the same cycle. Negligence ranks \( y \) head of \( z \) and \( x \) ahead of \( y \), and “equitable consideration” does not come into play with regard to either pair. It does come into consideration, and displaces, negligence as between \( x \) and \( z \), thus producing a cycle.

A more commonplace doctrine to take the place of either freedom of contract or the “equitable consideration” of the previous paragraph would be a fiduciary duty doctrine (such as might prevail between a captain and his passenger on a ship, or between most professionals and their clients), which typically prohibits a party from benefiting at the expense of another, even if that is cost-justified, in the sense that his benefits would exceed the other party’s loss. If we posit that kind of fiduciary relationship between \( x \) and \( z \), then it would operate in the same way to produce a cycle. In other words, negligence allows \( y \) to prevail over \( z \) and would allow \( x \) to prevail over \( y \). The fiduciary doctrine, however, would allow \( z \) to prevail over \( x \). If \( x \) is the decision maker we now have a fairly typical risk-creation scenario in which someone, namely \( x \), has to make a decision that will affect other parties and has to choose one among several feasible way of distributing risks among them. If such a scenario is subject to the negligence doctrine and some other doctrine that operates like freedom of contract, “equitable consideration,” or fiduciary duty doctrine, we get cycles of just the sort the theorem contemplates.

These examples are actually all structurally similar to a cycling problem long familiar to the law, but mistakenly thought to be somewhat exotic, namely the problem of circular priorities that can arise in property law and in the law of secured transactions. Owner \( O \) first sells his property to Buyer 1; next he fraudulently sells the same property to Buyer 2, and finally he sells it a third time to Buyer 3. In the end, the authorities have to decide who among the buyers has priority over whom. They will typically do so by resorting to several appealing doctrines, each of which taken by itself may be conditionally lex-utilitarian, but which in combination no longer are. The first doctrine provides that prior purchase prevails over subsequent purchases. A second doctrine provides that if a purchase files a record of his purchase in an official “record-book,” he prevails over one who did not, and that if purchase
filed, the first to file prevails. Finally, a third doctrine provides that if a later purchaser files his purchase in the record book, ahead of previous purchaser who did not, but in fact has notice of the prior purchase, he loses to the prior purchaser. A cycle arises in the case in which Buyer 1 buys, but does not file, Buyer 2 files but knows that Buyer 1 bought and Buyer 3 buys, files, but does not know about any of the prior buyers.

We are well aware that our result bears a close affinity to various well-known impossibility results from social choice theory, such as those of Arrow and Sen (see Miller and Rachmilevitch 2014 for a recent result in this literature). To be sure, their results involve the aggregation of different preferences, whereas our result involves the aggregation of different legal doctrines. However, the Legal Cycling Theorem links preferences and doctrines and so, results in social choice theory can be transposed into theorems about law. This is not to say, however, that their results are identical to ours, simply that they are in the same spirit, and served as the indispensable inspiration for ours. We now describe in detail the connection between the results in this paper and social choice theory.

5.2 Relationship to Social Choice Theory

As mentioned, this paper deals with legal systems. Typical social choice results do not. However, even at an abstract level there are differences between our results and social choice theory. To show them we now compare the Legal Cycling Theorem and the Combination Theorem with social choice theory.

The Legal Cycling Theorem takes a preference order $P$ (without indifference) and a legal system $L$, as input. The output is a choice function $C_{P,L}$. In the direction showing that lex-utilitarian legal system do not induce cycles, the Legal Cycling Theorem can be seen as a successful form of aggregation. A lex-utilitarian legal system has the logical structure of a preference order (with possible indifferences). So, two preferences orders (the one of the law abiding citizen and the one the legal system) are aggregated and the result is a new preference order $\succ$ determined by the choice function $C_{P,L}$ of the law abiding citizen. Given two options $x$ and $y$, $\succ$ ranks $x$ above $y$ (i.e., $x \succ y$) when $x$ is chosen over $y$. So,

$$x \succ y \iff C_{P,L}(x, y) = x.$$

This aggregation of preferences does not contradict any impossibility theorem in social choice (e.g., Arrow’s impossibility theorem). First, the (output) preference $\succ$ do not have the exact same logical structure as the (input) preferences determined by the legal system. The former does not allow for indifferences and the latter does. Moreover, $\succ$ does not directly oppose the legal system preferences. If $\succ$ ranks $x$ above $y$ then the legal system ranks $x$ either above or at the same level of $y$. 
In the converse direction, the Legal Cycling Theorem shows that any non lex-utilitarian legal system induces cycles. In this direction, the Legal Cycling Theorem is quite different from existing social choice theory. The typical input in the aggregation functions of social choice theory are preference orders. In the converse direction, the input in Legal Cycling Theorem is a complementary universe: all legal systems that are not preference orders.

At an abstract level, the Combination Theorem can be interpreted as an unusual result in social choice theory. It shows the impossibility of aggregating lex-utilitarian legal doctrines into a lex-utilitarian legal system. So, the output does not have the same logical structure as the input. The main difference is completeness (i.e., the need to determine the relative rank of alternatives). A legal system must express a viewpoint on which options are legal for any given issue. A lex-utilitarian legal system is associated with a transitive and complete binary relation. A doctrine need not express a viewpoint on the legality of different options in some issues. Hence, the binary relation associated with lex-utilitarian doctrines need not be complete.

In the classic impossibility theorems of Arrow and Sen, there are critical assumptions such as independence of irrelevant alternatives and liberalism. No such assumptions are made in this paper. The Combination Theorem relies only on the principle of unanimity. The Legal Cycling Theorem is a full characterization result that does not rely on standard desiderates of social choice. This is technical, but fundamental, difference between the results in this paper and the ones in social choice theory.

We expressed the preference \( P \) as the one of an individual: the law-abiding citizen. However, nothing prevents the preference \( P \) to be the one of a group or the entire society. Social choice theory often shows that individual preferences order cannot always be aggregated into a social preference order. The results in this paper show an additional difficulty. Even in the cases that society’s preferences can be expressed by an order (e.g., a social welfare function) social choices can be cyclic if they are required to be legal and the legal system is not lex-utilitarian.

6 Classic Results Revisited

As we have shown, cycles are not necessarily the result of irrationality, but may simply follow from the fact that plausible legal regimes are not lex-utilitarian. This new perspective on cycles makes it natural to revisit fundamental results obtained when cycles are ruled out. Here we focus on Sen’s liberal paradox and the results of Kaplow and Shavell on fairness versus welfare (Sen 1979; Kaplow and Shavell 2002).

(1) Sen’s liberal paradox.
Sen’s so-called liberal paradox reveals a conflict between a legal system’s granting its citizens rights of an even rudimentary nature and respecting the Pareto principle. If the system grants rights and if it respects the Pareto principle, it gives rise to a cycle. Sen’s claim has often been attacked on the ground that he had an implausible, eccentric conception of rights. While we don’t believe that criticism valid, there is no reason for us to address it here. Instead we note that our result casts the implications of Sen’s result in a rather different light. His result suggested that we must choose between rights and the Pareto principle. One can’t have both, he argued, since that leads to intransitivity. Now it should be evident that Sen’s result can be thought of as simply an instance of the type of intransitivity most rule-based systems generate. A commitment to both rights and the Pareto principle can lead to cycles just as the combination of two or more doctrines can produce non-lex-utilitarian regimes and non-lex-utilitarian regimes can induce cycles. However inasmuch as all plausible legal regimes produce such cycles, and live with them, we can say the same thing about Sen’s situation. We can have both elementary rights and the Pareto principle despite the fact that they lead to a cycle. The existence of a cycle in and of itself does not seem a compelling reason to rule either of them out of bounds, or to force a choice between them.

(2) Kaplow and Shavell on fairness versus welfare.

Kaplow and Shavell showed that if one combines the Pareto principle with any kind of fairness-based principles that will produce a cycle. They therefore argue that fairness-based legal principles should be rejected. However, if the Pareto principle is combined with any other doctrine, that will also produce a cycle. So, our result casts a different light on that implication in much the same way it does with respect to Sen. We can say what we say about Sen, simply replacing “rights” with “fairness-based principles:” That is, a commitment to both fairness-based principles and the Pareto principle ends up in a cycle just as all decision-makers in a non-lex-utilitarian regime end up in a cycle. However, inasmuch, as all plausible legal regimes produce such cycles, and live with them, we can say the same thing about Kaplow and Shavell’s situation. The existence of a cycle in and of itself does not seem a compelling reason to rule either of them out of bounds.

Other work, more broadly related to our themes, includes Spitzer 1979, Easterbrook 1982, Kornhauser and Sager 1986, and Cherepanov, Feddersen, and Sandroni 2013.

7 Legal Strategizing and the Concern with Cycles

Our argument that cycles are inevitable does not however mean that there are no legitimate concerns with cycles. It is generally understood that where there is cycling there are ample opportunities for strategic behavior—for manipulation. This is most familiar in the voting
context, in which control of the agenda, and especially the sequence in which certain issues are voted on, can greatly influence the outcome. Cycling in the legal context would seem to harbor those same possibilities. It is part of our research agenda to explore the various strategic opportunities produced by cycling in law. In a future paper, we will show that, just like in the voting context, cycles induced by law give law-abiding citizens opportunities to game the law. Hence, as a corollary of the results in this paper, the opportunities to manipulate the law are also ineradicable.

The analysis of legal strategizing produced by cycles is beyond the scope of this paper. So, we will simply draw the reader’s attention to the most immediately obvious one: By manipulating the order in which certain choices are made a great deal of what might look like circumvention of rules is made unavoidable. We know that in any cycle, it should be possible to end up where you want to end up, regardless of where you start out, so long as you make the right sequence of choices. Let us illustrate that with an artificial, but nonetheless illuminating example using the duress situation described above.

The original version of the duress example involved a defendant who is threatened with something very painful unless he helps the people who have made the threat commit some crime, for which he can validly claim the defense of duress. By contrast, we noted, if he had been threatened with the destruction of a treasured manuscript he has labored over for many years, and if, to avert the manuscript’s destruction, he had assisted them in their planned crime, he would not qualify for the defense. This gave rise to a cycle because the defendant is allowed to choose to endure great pain in exchange for protecting his manuscript, he is allowed to commit a serious crime so as to avoid being subjected to the painful treatment, but he is not allowed to commit the crime so as to prevent his manuscript from being destroyed. Here is how he might exploit this intransitivity strategically: Suppose the defendant is determined to do the equivalent of saving his manuscript by committing a crime. What he does is to pay off the people who are seeking to recruit him for a crime with money that he borrows from a loan shark. This loan shark in turn demands that he commit a serious crime as a way of extinguishing his debt, which he cannot pay. But if he committed a crime to escape the loan shark’s threats, he would most likely qualify for the defense of duress, because at this point he is doing so not to protect his manuscript but to avert great physical harm.

The example is of course contrived. But the contrivance is the sort that is bound to have more realistic counterparts. Wherever there is intransitivity, there is an opportunity, at least if the context is even mildly propitious, for strategic exploitation of this sort. The full scope of such opportunities, including the exploitation of “menu” effects and related phenomena, we plan to explore in another paper.

We now mention two more topics for future research just in passing. The first is legal
reasoning. Legal reasoning often depends on pointing out that if one commits to a series of principles they might clash, in the sense that they lead to cycles, and one of them should therefore be abandoned. Presumably some of these types arguments still pass muster, but others may now have to be rejected. It is a task for the future to figure out which arguments from intransitivity prove that a legal rule must be rejected and which do not. A second topic is that of rights. People are often very skeptical of the concept of rights ("nonsense of stilts" in Bentham’s famous phrase) because it leads to various kinds of intransitivities. What is known as the deontological paradox is perhaps the clearest example. These objections need to be rethought. Perhaps many of the objectionably paradoxical features of rights are simply paradoxical features that all legal regimes that are not lex-utilitarian exhibit.

8 Related Claims

We want to briefly say something about the relationship of our work to observations others have made about intransitivity in law. We cannot be exhaustive here but have to limit ourselves to results that seem closest to ours.

First, there is the work of Bruce Chapman, (Chapman 2003). While our thesis that rules inevitably breed cycles is an unfamiliar one, the Canadian scholar Bruce Chapman has maintained for many years now that intransitivity is a common feature of legal systems—not a blemish to be rooted out, but an essential feature to be explored and better understood. The rationality of legal decision making, he has argued, is different from the rationality of the utility-maximizing consumer or producer in the standard economic model. In a sense our paper is but following Chapman’s pioneering lead, seeking to more formally vindicate the intuition that has guided his work.

Second, there is the work of Larry Temkin, (Temkin 2001). In a recent path breaking book, Rethinking the Good, Temkin makes much of intransitivity in our everyday moral judgments, especially in what he calls “all things considered” judgments. An example of the kind of intransitivity he lays bare is the torture-hangnail paradox. Suppose we compare a person suffering a brief period of torture with the possibility of that person suffering a slightly less painful form of torture for a sufficiently longer period. If we just make sure that the slightly lesser pain lasts long enough, we would then probably consider the latter worse than the former: Less pain, but for a much longer time. Let’s repeat this step several times over, in each case slightly decreasing the amount of torture, and vastly increasing the amount of time spent enduring it. At some point the suffering inflicted will have diminished to the equivalent of a hangnail, albeit a hangnail that has to be endured for a very long time indeed. In other words, by this step-by-step process we have been driven to conclude
that a hangnail endured for a long time is worse than torture endured for a relatively brief (though far from evanescent) period. That seems absurd. This is Temkin’s exhibit 1 for a very widespread kind of intransitivity in what look like the kinds of “all things considered” judgments we make every day.

Note however that this intransitivity arises in a rather different manner than the one that we consider. It does not involve the interaction of rules with preference maximizing choices, or of rules with the Pareto principle, but rather involves a series of specific intuitively plausible comparative judgments that unexpectedly clash.

9 Extensions

9.1 Choices with more than two options

In this section, we extend our main results to choices with two or more options. So, in this section, let an issue $B$ now be a subset of $A$ with two or more elements, all of which different from each other. Unless otherwise stated, other definitions from previous sections remain valid. So, a legal system is still a mapping $\mathcal{L} : B \rightarrow B \cup A$ such that $\emptyset \neq \mathcal{L}(B) \subseteq B$. However, to differentiate the case of binary choice from the general case we refer to $\mathcal{L}$ as a full legal system if choices are not necessarily binary. The definition of what it means for a full legal system to be lex-utilitarian is a direct extension of our previous definition.

Definition 7 A full legal system $\mathcal{L}$ is lex-utilitarian if there is an utility function $u : A \rightarrow \mathbb{R}$ such that for every issue $B$,

$$x \in \mathcal{L}(B) \Leftrightarrow u(x) \geq u(y) \text{ for every } y \in B.$$ 

As before, a full legal system is lex-utilitarian if there is a function that ranks all theoretically possible alternatives and deems legal the top ones and only the top ones.

If choices may involve more than two options there are different types of behavior that are inconsistent with the choices of a rational agent subject to physical constraints. This type of behavior is known as WARP (the Weak Axiom of Revealed Preference) violations.

Formally

Definition 8 A choice function $C$ violates WARP if there are two issues $B$ and $B^*$ such that

$$B \subseteq B^*, \ C(B^*) \in B \text{ and } C(B) \neq C(B^*).$$

A violation of WARP occurs when the choice $C(B^*)$ in the super-set $B^*$ is in the sub-set $B$, but it is not chosen. A cycle implies a WARP violation because if $C(\{x, y\}) = x,$
\[ C(\{y, z\}) = y, \text{ and } C(\{x, y\}) = z \] then no matter which choice is made on the issue \( \{x, y, z\} \) there is a WARP violation. A choice function such as
\[ C(\{x, y\}) = x, \ C(\{y, z\}) = y, \ C(\{x, z\}) = x, \ C(\{x, y, z\}) = z \]
is not necessarily cyclic, but violates WARP because \( z \) is rejected against \( x \) or \( y \), but \( z \) is chosen against both \( x \) and \( y \).

**Definition 9** A full legal system \( \mathcal{L} \) induces WARP violations if there exists a preference \( P \) such that the resulting choice function of a law abiding citizen \( C_{P,\mathcal{L}} \) violates WARP.

We also speak of a legal system inducing WARP violations because if there were no law, there would be no violations of WARP (Samuelson 1938).

**Extended Legal Cycling Theorem** Consider the case where there are at least three distinct alternatives. Then,

(a) No full legal system that is lex-utilitarian induces WARP violations.

However,

(b) Any full legal system that is not lex-utilitarian induces WARP violations.

**Proof:** See Appendix.

The Extended Legal Cycling Theorem characterizes the full legal systems that induce WARP violations. It is the counterpart of the Legal Cycling Theorem when choices involve two or more alternatives. The Combination Theorem is an impossibility result that holds in the case of binary choice and, therefore, it also holds in the case of choices involving two or more options as well.

### 9.2 Allowing Indifference

So far, we have not allowed the law abiding citizen to be indifferent between options. In this section, we show a counterpart of the Legal Cyclic theorem that holds even if indifference is allowed. As before we make changes in some, but not all, definitions.

In this section, a preference \( P \) is a complete, transitive binary relation (i.e., an order). So, \( P \) may or may not be asymmetric. In principle, there may exist two distinct alternatives \( x \) and \( y, x \neq y \), such that
\[ x \ P \ y \text{ and } y \ P \ x. \]

This is the case of indifference between \( x \) and \( y \). Indifference makes it possible for more than one option be optimal and, therefore, for more than one option to be selected. A choice
correspondence $C$ is a mapping $C : B \to B \cup A$ such that $C(B) \subseteq B$. The law abiding citizen choice correspondence $C$ ($= C_{P,L}$) is such that

$$C_{P,L}(B) \subseteq L(B) \text{ and } C_{P,L}(B) P y \text{ for every } y \in L(B), y \neq C_{P,L}(B).$$

Hence, as before, $C_{P,L}(B)$ optimizes $P$ on $L(B)$. However, $C_{P,L}(B)$ may contain more than one option. These are the (perhaps multiple) options that the law abiding citizen prefer among the legal ones. Consider the following example: A preference $P^*$ is indifferent between $x$, $y$, and $z$. Assume no law and so, $L^*(B) = B$ for every issue $B$. Then, $C^*$ ($= C_{P^*,L^*}$) is such that

$$C^*(\{x,y\}) = \{x,y\}, \ C^*(\{y,z\}) = \{y,z\} \text{ and } C^*(\{x,z\}) = \{x,z\}.$$

So, a choice correspondence where all options are selected is permitted, even in the case of a standard economic agent that optimizes a preference $P$ on $B$.

Now consider the choice correspondence $\bar{C}$,

$$\bar{C}(\{x,y\}) = \{x\}, \ \bar{C}(\{y,z\}) = \{y,z\}, \text{ and } \bar{C}(\{x,z\}) = \{x,z\}.$$

This choice correspondence is not possible for a standard economic agent. In the absence of any legal restriction, $\bar{C}(\{x,y\}) = \{x\}$ implies a strict preference for $x$ over $y$, while $\bar{C}(\{y,z\}) = \{y,z\}$ and $\bar{C}(\{x,z\}) = \{x,z\}$ implies indifference between $x$, $y$, and $z$. This is an example of a choice correspondence that we refer to as non-spuriously cyclic. More generally,

**Definition 10** A choice correspondence $C$ is **non-spuriously cyclic** if there exist three distinct alternatives $x$, $y$ and $z$ such that

$$C(\{x,y\}) = \{x\}; \ y \in C(\{y,z\}); \ z \in C(\{x,z\}).$$

Non-spuriously cyclic choice correspondence are those that may induce cycles and cannot be produced by optimal choice when there is no law. Naturally, to be (or not) non-spuriously cyclic is a property of the choice correspondence itself. It is not a property of final selection that might be made among optimal options.

**Definition 11** A legal system $L$ **induces non-spurious cyclic choice correspondences** if there exists a preference $P$ such that the resulting choice correspondence $C_{P,L}$ of a law abiding citizen is non-spuriously cyclic.

We also speak of a legal system **inducing** non-spurious cyclic choice correspondences because they are not possible if there is no law and all options are legal.
Legal Cycling Theorem (with possible indifferences) Consider the case where issues are binary choices and there are at least three distinct alternatives. Then,

(a) No lex-utilitarian legal system induces non-spurious cyclic choice correspondences.

However,

(b) Any legal system that is not lex-utilitarian induces non-spurious cyclic choice correspondences.

Proof: See Appendix.

The Legal Cycling Theorem (with possible indifferences) characterizes the legal systems that induce non-spurious cyclic choice correspondences. In this variation of the Legal Cycling Theorem, the law-abiding citizen may be indifferent between options.

In the Combination Theorem the law itself is cyclic (i.e., CL holds). Then, the choices of a law abiding citizen are cyclic regardless of preferences. In particular, it does not matter whether the law abiding citizen may or may not be indifferent between options.

Finally, the Legal Cycling Theorem can also be modified to accommodate indifference and choices involving two or more options, but cycles must be replaced with the more general phenomena of WARP violations.

Definition 12 A choice correspondence $C$ non-spuriously violates WARP if there exist two issues $B$ and $B^*$ and an option $y$ such that

$$B \subseteq B^*, \ y \in C(B^*) \cap B \text{ and } y \notin C(B).$$

This definition is the counterpart of WARP violations for correspondences. Any choice function that violates WARP is also a choice correspondence that non-spuriously violates WARP.

Definition 13 A full legal system $L$ induces choice correspondences that non-spuriously violate WARP if there exists a preference $P$ such that the resulting choice correspondence of a law abiding citizen $C_{P,L}$ non-spuriously violates WARP.

That is, consider a full legal system induces choice correspondences that non-spuriously violate WARP. They produce choice correspondences that do not arise in the absence of law.

Extended Legal Cycling Theorem (with possible indifferences) Consider the case where there are at least three distinct alternatives. Then,

(a) No full legal system that is lex-utilitarian induces choice correspondences that non-spuriously violate WARP.
However,

(b) Any full legal system that is not lex-utilitarian induces choice correspondences that
non-spuriously violate WARP.

**Proof:** See Appendix.

This result shows that even if the law-abiding citizen may be indifferent between options
then the types of choice correspondence that may occur in the absence of any law (i.e., by
standard economic agents) are the ones that arise under lex-utilitarian full legal systems and
only by them. If the full legal system is not lex-utilitarian then it induces correspondences
with a logical structure that do not arise in the absence of law.

### 9.3 Context-dependent Alternatives

Alternatives are required to be independent of their context. That is, the meaning of an
option $x$ does not change depending on whether $y$ or $z$ is also available. This assumption
is implicit in almost every formal model and in this one as well. As we now show, if this
assumption is relaxed then the decision maker cannot order all alternatives because some
choices would be impossible to make.

To see this, recall that a preference order is a complete, transitive binary relation. These
are the two traditional pillars of rationality. Complete and transitive preferences are often
referred to as rational (see definition 1.B.1, chapter 1 in Microeconomic Theory; Mas Colell,
Whiston, and Green 1995). Completeness requires some decision to be possible between
two any alternatives. Now for concreteness consider option $x$ (to use deadly force) in our
self-defense example in section 3.3. Let’s break option $x$ into two new alternatives: $x_l$ and
$x_i$. Option $x_i$ is to use deadly force legally (i.e., in self-defense). Option $x_i$ is to use deadly
force illegally. In this example, the use of deadly force is legal if the alternative is to incur a
serious injury ($y$), but not if it is to have the manuscript damaged ($z$). Thus, it is impossible
to make a choice between $x_l$ and $z$. It is also impossible to make a choice between $x_i$ and $y$
for the same reason. If $z$ is the available alternative then the use of deadly force is illegal and,
hence, $x_l$ is not available. So, one cannot make a choice between $x_l$ and $z$ because $x_l$ does
not exist in the presence of $z$. Naturally, this is a general phenomena that does not hinge on
the specific example concerning self-defense. If an option ceases to exist in the presence of
another option than completeness no longer holds and, therefore, the logical structure of the
choice function is not the same as in traditional choice theory that abstracts away from legal
constraints. This limitation, however, does not apply to context-independent contingencies.
For example “taking an umbrella if it rains” is context-independent because whether or not
it rains does not depend on the available alternatives.
10 Conclusion

A legal system is lex-utilitarian if it is possible to rank-order all legal options a citizen might face, and if the system requires that he chooses the highest ranked alternative among the options available to him. Our main result here is a theorem showing that a lex-utilitarian system is the only one that can avoid cycling. We then show, through suitably representative examples and one general proposition, why no acceptable legal system is lex-utilitarian, and why all acceptable systems are therefore bound to induce cycles. Several implications of this result remain to be explored.

11 Appendix

11.1 Proof of the Legal Cycling Theorem (and Variations)

First consider the basic case where issues are binary choices and preferences are asymmetrical orders. That is, we first demonstrate the legal cycling theorem.

Assume that $L$ is a lex-utilitarian legal system. Also assume, by contradiction, that there is an asymmetric preference order $P$ such that, for the resulting choice function $C (= C_{P,L})$, there are distinct alternatives $x$, $y$ and $z$ such that $C(\{x,y\}) = x; C(\{y,z\}) = y; C(\{x,z\}) = z$. Then, $x \in L(\{x,y\})$, $y \in L(\{y,z\})$, and $z \in L(\{x,z\})$. So, $u(x) \geq u(y) \geq u(z) \geq u(x)$. Thus, $u(x) = u(y) = u(z)$. It now follows that $L(\{x,y\}) = \{x,y\}$, $L(\{y,z\}) = \{y,z\}$, and $L(\{x,z\}) = \{x,z\}$. Thus, $xP y$ and $yP z$ and $zP x$. A contradiction (with the transitivity of $P$).

Now for the converse. Assume, by contradiction, that $L$ is a legal system that is not lex-utilitarian and, for no asymmetric preference order $P$, the resulting choice function $C (= C_{P,L})$ is cyclic.

Step 1. There cannot be three distinct options $x$, $y$ and $z$ such that for two different pairs of options, say $\{x,y\}$ and $\{y,z\}$, the law allows both choices, e.g., $L(\{x,y\}) = \{x,y\}$ and $L(\{y,z\}) = \{y,z\}$, and for the remaining pair $\{x,z\}$, $L(\{x,z\})$ has only one element.

Assume that $L(\{x,z\}) = \{z\}$. Consider any asymmetric preference order such that $xP yP z$. Then, $C(\{x,y\}) = x; C(\{y,z\}) = y; C(\{x,z\}) = z$. Now assume that $L(\{x,z\}) = \{x\}$. Consider any asymmetric preference order such that $zP yP x$. Then, $C(\{z,y\}) = z; C(\{y,x\}) = y; C(\{x,z\}) = x$.

Let $\succ$ be the binary relation defined by

$$x \succ y \iff L(\{x,y\}) = \{x\}.$$

Step 2 $\succ$ is transitive.
Assume that $x \succ y \succ z$. If $\mathcal{L}(\{x, z\}) = \{z\}$ then, for any preference $P$, the choices of a law abiding citizen are cyclic (because the law requires $x$ to be chosen over, $y$ over $z$ and $z$ over $x$). If $\mathcal{L}(\{x, z\}) = \{x, z\}$ then consider any asymmetric preference order $P$ such that $z \succ P x$, the choices of a law abiding citizen are cyclic. Thus, $x \succ z$.

Let $S$ a chain be a sequence of options $x_n, ..., x_1$ such that $x_j \succ x_i$ if $j > i$. Such chain must exist, otherwise all options are legal and so, $\mathcal{L}$ is a lex-utilitarian legal system. Moreover, given that $A$ is finite, there must exist a longest chain (one for which the number of elements in it is maximal). With some abuse of notation, let $S = [x_n, ..., x_1]$ be a longest, not necessarily unique chain. By definition, there is no option $x$ such that $x \succ x_n$ and no option $y$ such that $x_1 \succ y$.

Step 3 For any alternative $y$ that does not belong to the chain $S$, there exists a unique element $x_i \in S$ such that $\mathcal{L}(\{y, x_i\}) = \{y, x_i\}$.

Assume that there are two distinct elements, $x_i \in S$ and $x_j \in S$ such that $\mathcal{L}(\{y, x_i\}) = \{y, x_i\}$ and $\mathcal{L}(\{y, x_j\}) = \{y, x_j\}$. By definition, $\mathcal{L}(\{x_i, x_j\})$ has only one element. This contradicts step 1. Now assume that for some alternative $y$ there is no element $x_i$ in chain $S$ such that $\mathcal{L}(\{y, x_i\}) = \{y, x_i\}$. Thus, $x_n \succ y$ and $y \succ x_1$. The chain $S$ must contain more than 2 options. Otherwise the chain $x_n \succ y \succ x_1$ is longer. Moreover, for any $i = 2, ..., n$, if $x_i \succ y$ then $x_{i-1} \succ y$. Otherwise $x_i \succ y \succ x_{i-1}$ and so, the chain $x_n \succ ...x_i \succ y \succ x_{i-1} \succ ... \succ x_1$ is longer. It follows that either $x_1 \succ y$ or $y \succ x_n$. A contradiction.

Given any alternative $y$ not in the chain $S$, let $i(y) \in \{1, ..., n\}$ be such that $\mathcal{L}(\{y, x_{i(y)}\}) = \{y, x_{i(y)}\}$. If $z$ is in the chain $S$ let $i(z) \in \{1, ..., n\}$ be such that $z = x_{i(z)}$. For any alternative $x$, let $u(x) = i(x)$. By step 3, $u$ is well defined.

Step 4. If an option $y$ is not in the chain $S$ then $\mathcal{L}(\{y, x_j\}) = \{y\}$ if $j < i(y)$ and $\mathcal{L}(\{y, x_j\}) = \{x_j\}$ if $j > i(y)$.

The case $\mathcal{L}(\{y, x_j\}) = \{y, x_j\}$ can be ruled out by step 1 because $\mathcal{L}(\{x_j, x_{i(y)}\})$ has only one element and, by definition, $\mathcal{L}(\{y, x_{i(y)}\}) = \{y, x_{i(y)}\}$. Let $x_i = x_{i(y)}$. Assume that $j < i(y)$. Then, $C(\{x_i, x_j\}) = x_i$. Consider any asymmetric order $P$ such that $y \succ P x_i$. Then, $C(\{x_i, y\}) = y$. If $\mathcal{L}(\{y, x_j\}) = \{x_j\}$ then $C(\{y, x_j\}) = \{x_j\}$. This forms a cycle. So, $\mathcal{L}(\{y, x_j\}) = \{y\}$. Now assume that $j > i$. Then, $C(\{x_i, x_j\}) = x_j$. Consider any order $P$ such that $x_i \succ P y$. Then, $C(\{x_i, y\}) = x_i$. If $\mathcal{L}(\{y, x_j\}) = \{y\}$ then $C(\{y, x_j\}) = y$. This forms a cycle. So, $\mathcal{L}(\{y, x_j\}) = x_j$.

Step 5. If $z$ and $y$ are distinct options and neither is in $S$ then $\mathcal{L}(\{y, z\}) = z$ if $i(y) < i(z)$; $\mathcal{L}(\{y, z\}) = y$ if $i(y) > i(z)$ and $\mathcal{L}(\{y, z\}) = \{y, z\}$ if $i(y) = i(z)$.

Consider the case $i(y) < i(z)$. By step 4, $\mathcal{L}(\{z, x_{i(y)}\}) = z$. By definition, $\mathcal{L}(\{y, x_{i(y)}\}) = \{y, x_{i(y)}\}$. By step 1, the case $\mathcal{L}(\{y, z\}) = (y, z)$ can be ruled out. Now consider the case $\mathcal{L}(\{y, z\}) = y$. Then, $C(\{y, z\}) = y$. Given that $\mathcal{L}(\{z, x_{i(y)}\}) = z$ then $C(\{z, x_{i(y)}\}) = z$. Now consider any order $P$ such that $y \succ P x_{i(y)}$. By definition, $\mathcal{L}(\{y, x_{i(y)}\}) = \{y, x_{i(y)}\}$. So,
C(\{y, x_{i(y)}\}) = y. This forms a cycle. Thus, \(L(\{y, z\}) = z\). The proof on the case \(i(y) > i(z)\) is the same with just a change in labels. Hence, it is omitted. Now consider the case \(i(y) = i(z)\). Let \(x_i = x_{i(y)} = x_{i(y)}\). By definition, \(L(\{y, x_i\}) = \{y, x_i\}\) and \(L(\{z, x_i\}) = \{z, x_i\}\). So, by step 1, \(L(\{y, z\}) = \{y, z\}\).

The proof is now concluded as follows: Let \(x\) and \(y\) be the distinct options. If both of them belong to the chain \(S\) then, by definition, \(u(x) \neq u(y)\) and \(L(\{x, y\}) = x\) if \(u(x) > u(y)\). If one of them, say \(x\) belongs to the chain \(S\), and the other, \(y\), does not then, by step 4, \(L(\{x, y\}) = x\) if \(u(x) > u(y)\) and \(L(\{x, y\}) = y\) if \(u(y) > u(x)\). By definition, \(L(\{x, y\}) = \{x, y\}\) if \(u(y) = u(x)\). If both \(x\) and \(y\) do not belong to \(S\) then, by step 5, \(L(\{x, y\}) = \{x\}\) if \(u(x) > u(y)\), \(L(\{x, y\}) = \{y\}\) if \(u(y) > u(x)\) and \(L(\{x, y\}) = \{x, y\}\) if \(u(y) = u(x)\). Therefore, \(L\) is lex-utilitarian. A contradiction. This demonstrates the legal cycling theorem.

Now we consider the general case where issues can have two or more alternatives in it. So, we now demonstrate the extended legal cycling theorem.

Assume that \(L\) is a lex-utilitarian full legal system. Also assume, by contradiction, that there is an asymmetric preference order \(P\) such that, for the resulting choice function \(C (= C_{P,L})\), there are issues \(B\) and \(B^*\) such that \(B \subseteq B^*\), \(C(B^*) \in B\) and \(C(B) \neq C(B^*)\). Then, \(u(C(B)) \geq u(C(B^*))\) (because \(C(B^*) \in B\)) and \(u(C(B^*)) \geq u(C(B))\) (because \(C(B) \in B^*\)). Thus, \(u(C(B)) = u(C(B^*))\). Therefore, \(C(B) \in L(B^*)\) and \(C(B^*) \in L(B)\). It follows that \(C(B) \in P C(B^*)\) and \(C(B^*) \in P C(B)\). A contradiction.

Now for the converse. Assume, by contradiction, that \(L\) is a full legal system that is not lex-utilitarian and, for no asymmetric preference order \(P\), the resulting choice function \(C (= C_{P,L})\) violates WARP. Then, in particular, no resulting choice function \(C\) is cyclic. Hence, by the argument above, the utility function \(u\) is well defined and such that

\[L(\{x, y\}) = \{x, y\}\) if \(u(x) = u(y)\); \(L(\{x, y\}) = \{x\}\) if \(u(x) > u(y)\).

Now assume that \(x \in L(B)\) and \(u(y) > u(x)\) for some \(y \in B\). Then, \(L(\{x, y\}) = y\). Let \(P\) be any preference order such that \(x \triangleright_P z\) for any \(z \neq x\). Then, \(C(B) = x\) and \(C(x, y) = y\). Thus, \(C\) violates WARP. A contradiction. Now assume that \(u(x) \geq u(y)\) for every \(y \in B\) and \(x \notin L(B)\). Then, \(C(B) \neq x\) and \(u(x) \geq u(C(B))\). Let \(z = C(B)\). So, \(x \in L(\{z, x\})\). Let \(P\) be any asymmetric preference order such that \(x \triangleright_P z\). Then, \(x = C(x, z), x \in B\) and \(z = C(B) \neq x\). Thus, \(C\) violates WARP. A contradiction.

Now we demonstrate the legal cycling theorem (with possible indifferences). Assume that \(L\) is a lex-utilitarian legal system. Also assume, by contradiction, that there is a preference order \(P\) such that, for the resulting choice function \(C (= C_{P,L})\), there are distinct alternatives \(x, y\) and \(z\) such that

\[C(\{x, y\}) = \{x\}; y \in C(\{y, z\}); z \in C(\{x, z\})\].
Then, \( x \in \mathcal{L}(\{x, y\}) \), \( y \in \mathcal{L}(\{y, z\}) \), and \( z \in \mathcal{L}(\{x, z\}) \). So, \( u(x) \geq u(y) \geq u(z) \geq u(x) \). Thus, \( u(x) = u(y) = u(z) \). It now follows that \( \mathcal{L}(\{x, y\}) = \{x, y\} \), \( \mathcal{L}(\{y, z\}) = \{y, z\} \), and \( \mathcal{L}(\{x, z\}) = \{x, z\} \). Thus, \( x P y \) and \( y P z \) and \( z P x \). By transitivity, \( x P y \) and \( y P x \). Therefore, \( C(\{x, y\}) = \{x, y\} \). A contradiction.

For the converse. If \( \mathcal{L} \) is a not a lex-utilitarian legal system then there is a preference order (asymmetric) \( P \) such that the resulting choice function \( C (= C_{P, \mathcal{L}}) \) is cyclic. Therefore, \( C(\{x, y\}) = x; C(\{y, z\}) = y; C(\{x, z\}) = z \). It follows that \( C \) is non-spuriously cyclic.

Finally, we demonstrate the extended legal cycling theorem (with possible indifferences). Assume that \( \mathcal{L} \) is a lex-utilitarian full legal system. Also assume, by contradiction, that there is a preference order \( P \) such that, for the resulting choice function \( C (= C_{P, \mathcal{L}}) \), there are issues \( B \) and \( B^* \) and an option \( y \) such that

\[
B \subseteq B^*, \quad y \in C(B^*) \cap B \text{ and } y \notin C(B).
\]

Let \( z \in \mathcal{L}(B) \). Then, \( u(z) \geq u(y) \) (because \( y \in B \)) and \( u(y) \geq u(z) \) (because \( y \in \mathcal{L}(B^*) \)) and \( z \in B^* \) given that \( B^* \supseteq B \supseteq \mathcal{L}(B) \). Thus, \( u(y) = u(z) \). Therefore, \( z \in \mathcal{L}(B^*) \). It follows that \( y P z \). This holds for any \( z \in \mathcal{L}(B) \). Hence, \( y \in C(B) \). A contradiction.

For the converse. If \( \mathcal{L} \) is a full legal system that is not a lex-utilitarian then there is a preference order (asymmetric) \( P \) such that the resulting choice function \( C (= C_{P, \mathcal{L}}) \) violates WARP. Thus, \( C \) non-spuriously violates WARP.

### 11.2 Proof of the Proposition

First consider the case \( n = 2 \). Let \( x, y \) and \( z \) be three different choices. Let \( u_1 : A \rightarrow \mathcal{R} \) be any function such as \( u_1(y) = u_1(z) > u_1(x) \). Let \( u_2 : A \rightarrow \mathcal{R} \) be an function such as \( u_2(y) = u_2(z) = u_2(x) \). Let \( \mathcal{D}_1 \) be any doctrine such that

\[
\mathcal{D}_1(\{y, z\}) = \{y, z\}; \quad \mathcal{D}_1(\{x, y\}) = n/a; \quad \mathcal{D}_1(\{x, z\}) = \{z\}.
\]

and for any other issue \( B = \{w, v\} \), where either \( w \notin \{x, y, z\} \) or \( v \notin \{x, y, z\} \) or both \( w \) and \( v \) do not belong to \( \{x, y, z\} \),

- either \( \mathcal{D}_1(\{w, v\}) = n/a \) or \( \mathcal{D}_1(\{w, v\}) = \{w, v\} \) if \( u_1(w) = u_1(v) \);
- either \( \mathcal{D}_1(\{w, v\}) = n/a \) or \( \mathcal{D}_1(\{w, v\}) = \{w\} \) if \( u_1(w) > u_1(v) \);
- either \( \mathcal{D}_1(\{w, v\}) = n/a \) or \( \mathcal{D}_1(\{w, v\}) = \{v\} \) if \( u_1(v) > u_1(w) \).

Let \( \mathcal{D}_2 \) be any doctrine such that

\[
\mathcal{D}_2(y, z) = (y, z); \quad \mathcal{D}_1(x, y) = (x, y); \quad \mathcal{D}_1(x, z) = n/a.
\]

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and for any other issue \( B = (w, v) \), where either \( w \notin \{x, y, z\} \) or \( v \notin \{x, y, z\} \) or both \( w \) and \( v \) do not belong to \( \{x, y, z\} \),

- either \( D_2(\{w, v\}) = n/a \) or \( D_2(\{w, v\}) = \{w, v\} \) if \( u_2(w) = u_2(v) \);

- either \( D_2(\{w, v\}) = n/a \) or \( D_2(\{w, v\}) = \{w\} \) if \( u_2(w) > u_2(v) \);

- either \( D_2(\{w, v\}) = n/a \) or \( D_2(\{w, v\}) = \{v\} \) if \( u_2(v) > u_2(w) \).

By construction, \( D_1 \) and \( D_2 \) are conditionally lex-utilitarian. Let \( \alpha \) be an aggregator that maps conditionally lex-utilitarian doctrines into lex-utilitarian legal systems and \( L = \alpha(D_1, D_2) \). By unanimity,

\[
L(\{y, z\}) = \{y, z\}; \quad L(\{x, y\}) = \{x, y\}; \quad L(\{x, z\}) = \{z\}.
\]

Thus, \( L \) is not lex-utilitarian. A contradiction.

The case \( n > 2 \) can be shown in exactly the same way.

The argument above shows our result in the case that \( L \) is not lex-utilitarian because \( (N) \) holds. A proof where \( L \) is not lex-utilitarian because \( (CL) \) holds can also be obtained. Again we focus on the case \( n = 2 \).

Let \( u_1 : A \longrightarrow R \) be any function such as \( u_1(x) > u_1(y) > u_1(z) \). Let \( u_2 : A \longrightarrow R \) be an function such as \( u_2(z) > u_2(x) \). Let \( D_1 \) be any doctrine such that

\[
D_1(\{x, y\}) = \{x\}; \quad D_1(\{y, z\}) = \{y\}; \quad D_1(\{x, z\}) = n/a
\]

and for any other issue \( B = (w, v) \), where either \( w \notin \{x, y, z\} \) or \( v \notin \{x, y, z\} \) or both \( w \) and \( v \) do not belong to \( \{x, y, z\} \),

- either \( D_1(\{w, v\}) = n/a \) or \( D_1(\{w, v\}) = \{w, v\} \) if \( u_1(w) = u_1(v) \);

- either \( D_1(\{w, v\}) = n/a \) or \( D_1(\{w, v\}) = \{w\} \) if \( u_1(w) > u_1(v) \);

- either \( D_1(\{w, v\}) = n/a \) or \( D_1(\{w, v\}) = \{v\} \) if \( u_1(v) > u_1(w) \).

Let \( D_2 \) be any doctrine such that

\[
D_1(\{x, y\}) = n/a; \quad D_2(\{y, z\}) = n/a; \quad D_1(\{x, z\}) = \{z\}.
\]

and for any other issue \( B = (w, v) \), where either \( w \notin \{x, y, z\} \) or \( v \notin \{x, y, z\} \) or both \( w \) and \( v \) do not belong to \( \{x, y, z\} \),

- either \( D_2(\{w, v\}) = n/a \) or \( D_2(\{w, v\}) = \{w, v\} \) if \( u_2(w) = u_2(v) \);

- either \( D_2(\{w, v\}) = n/a \) or \( D_2(\{w, v\}) = \{w\} \) if \( u_2(w) > u_2(v) \);
either $D_2(\{w, v\}) = n/a$ or $D_2(\{w, v\}) = \{v\}$ if $u_2(v) > u_2(w)$.

By construction, $D_1$ and $D_2$ are conditionally lex-utilitarian. Let $\alpha$ be an aggregator that maps conditionally lex-utilitarian doctrines into lex-utilitarian legal systems and $L = \alpha(D_1, D_2)$. By unanimity,

$$L(\{x, y\}) = \{x\}; L(\{y, z\}) = \{y\}; L(\{x, z\}) = \{z\}.$$  

Thus, $L$ is not lex-utilitarian.

References


